№4. In triangle ABC, a point M is the midpoint of AB, and a point I is the incentre. Point A_1 is the reflection of A in BI, and B_1 is the reflection of B in AI. Let N be the midpoint of A_1B_1. Prove that $IN > IM$.

First solution. Due to symmetry, we get $IA_1 = IA$ и $IB_1 = IB$. Therefore,

$$\begin{aligned}4(IN^2 - IM^2) &= \left|\overrightarrow{IA_1} + \overrightarrow{IB_1}\right|^2 - \left|\overrightarrow{IA} + \overrightarrow{IB}\right|^2 \\ &= \left(IA_1^2 + IB_1^2 + 2IA_1 \cdot IB_1 \cdot \cos\angle A_1IB_1\right) - \left(IA^2 + IB^2 + 2IA \cdot IB \cdot \cos\angle AIB\right) \\ &= 2IA \cdot IB \cdot (\cos\angle A_1IB_1 - \cos\angle AIB).\end{aligned}$$

So, to prove the required inequality $IN > IM$, it suffices to show that $\cos\angle A_1IB_1 > \cos\angle AIB$.

Notice that $\phi = \angle AIB = 90^\circ + \angle ACB/2 > 90^\circ$. By symmetry again, we have $\angle A_1IB = \angle AIB = \angle AIB_1 = \phi$. Therefore, if $\phi \le 120^\circ$, then

$$\angle A_1IB_1 = 360^\circ - (\angle A_1IB + \angle AIB + \angle AIB_1) = 360^\circ - 3\phi \in [0^\circ, \phi),$$

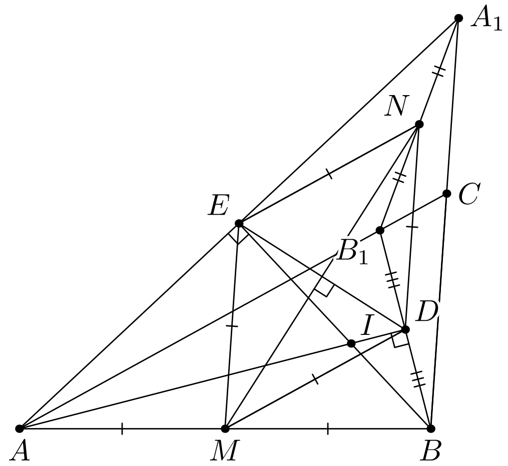
since $\phi > 90^\circ$. This yields the desired inequality.

Otherwise, if $\phi > 120^\circ$, then

$$\angle A_1IB_1 = (\angle A_1IB + \angle AIB + \angle AIB_1) - 360^\circ = 3\phi - 360^\circ \in (0^\circ, \phi),$$

since $\phi < 180^\circ$; this again yields the desired inequality.

Second solution. Notice that the angle AIB is obtuse, since $\angle AIB = 90^\circ + \angle ACB/2$. Clearly, A_1 lies on the line BC, while B_1 lies on the line AC. Let D and E denote the midpoints of the base sides BB_1 and AA_1 in the isosceles triangles BAB_1 and ABA_1, respectively. Then $\angle ADB = \angle AEB = 90^\circ$, which means that AE and BD are altitudes in the obtuse triangle AIB. Hence, the points I and M share the same side of the line DE. Moreover, the points A, B, D, and E lie on a circle centered at M.



The properties of a midline yield $DN = BA_1/2 = BA/2 = DM$, so that $DN = DM = AB/2$. Similarly, we obtain $EN = EM = AB/2$. Consequently, the quadrilateral $MDNE$ is a rhombus, in which the line DE is the perpendicular bisector of the diagonal MN. Since I and M share the same side of that line, we have $IM < IN$. (Indeed, the semiplane of the perpendicular bisector DE containing M is the locus of the points which are closer to M than to N. Since I lies in that halfplane, we have $IM < IN$.)

№5. A polynomial $f(x)$ with real coefficients of degree greater than 1 is given. Prove that there are infinitely many positive integers which cannot be represented in the form

$$f(n+1)+f(n+2)+\cdots+f(n+k)$$

where n and k are positive integers.

Solution.

Let the leading term of $f(x)$ be ax^m. If $a<0$, the number of integer x with positive $f(x)$ is finite, therefore sum $f(n+1)+\cdots+f(n+k)$ is bounded and has finitely many positive values. Thus we can confine ourselves to the case $a>0$. In this case $f(x)$ takes finitely many negative values for positive integer x, and there is some d such that $f(x+1)+f(x+2)+\cdots+f(x+d)$ is always positive.

Lemma. If $P(x)$ is a polynomial of degree m with positive leading coefficient, then $P(x)>bx^m$ for some positive b and all x greater than some C.

Indeed, if r is the leading coefficient of P, for each $b<r$ the polynomial $P(x)-bx^m$ has positive leading coefficient and is positive for lare enough x.

Polynomials $f(\frac{x}{2}-1)$ and $f(x)$ have the same degree m, therefore there exists $b>0$ such that $f(x)>$ $>bx^m$, and $f(\frac{x}{2}-1)>bx^m$ for $x>C$.

Let us consider large enough M and evaluate the number of pairs (n,k) such that $f(n+1)+\cdots+$ $+f(n+k)\leq M$.

If $n>\sqrt{\frac{M}{b}}$ (we take M large enough for the right-hand side to be greater than C), each term in the sum is greater than $bn^m\geq bn^2>M$, thus the sum is greater than M.

If $k>\sqrt[3]{\frac{2M}{b}}$ (we take M large enough for the right-hand side to be greater than $2d$), at least $k/2$ among the numbers $n+1,\ldots,n+k$ are no less than $k/2-1$, therefore, the respective terms are greater than $bk^m\geq bk^2$, and their sum is greater than $\frac{k}{2}\times bk^2=\frac{bk^3}{2}>M$. The rest of the sum is positive (since $k>2d$), and the entire sum is again greater than M.

Hende the number of pairs (n,k) of positive integers such that $f(n+1)+\cdots+f(n+k)\leq M$ does not exceed $\sqrt[3]{\frac{2}{b}}\cdot\sqrt{\frac{1}{b}}\cdot M^{5/6}$, which is less than $M/2$ for large enough M. We see that there are at least $M/2$ positive integers without desired representation, and M can be arbitrarily large.

№6. Do there exist two bounded sequences $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ such that for each positive integers n and $m > n$ at least one of the two inequalities $|a_m - a_n| > \frac{1}{\sqrt{n}}$, $|b_m - b_n| > \frac{1}{\sqrt{n}}$ holds?

Solution. Suppose such sequences (a_n) and (b_n) exist. For each pair (x, y) of real numbers we consider the corresponding point (x, y) in the coordinate plane. Let P_n for each n denote the point (a_n, b_n). The condition in the problem requires that the square $\{(x, y) : \ |x - a_n| \le \frac{1}{\sqrt{n}},\ |y - b_n| \le \frac{1}{\sqrt{n}}\}$ does not contain P_m for $m \ne n$.

For each point A_n we construct its *private square* $\{(x, y) : \ |x - a_n| \le \frac{1}{2\sqrt{n}},\ |y - b_n| \le \frac{1}{2\sqrt{n}}\}$. The condition implies that private squares of points A_n and A_m are disjoint when $m \ne n$.

Let $|a_n| < C$, $|b_n| < C$ for all n. Then all private squares of points A_n lie in the square $\{(x, y) : \ |x| \le C + \frac{1}{2},\ |y| \le C + \frac{1}{2}\}$ with area $(2C + 1)^2$. However private squares do not intersect, and the private square of P_n has area $\frac{1}{n}$. The series $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ diverges; in particular, it contains some finite number of terms with sum greater than $(2C + 1)^2$, which is impossible if the respetive private square lie inside a square with area $(2C + 1)^2$ and do not intersect. This contradiction shows that the desired sequences (a_n) and (b_n) do not exist.