№1. Non-zero polynomials $P(x)$, $Q(x)$, and $R(x)$ with real coefficients satisfy the identities

$$P(x) + Q(x) + R(x) = P(Q(x)) + Q(R(x)) + R(P(x)) = 0.$$

Prove that the degrees of the three polynomials are all even.

Solution. Let n be the largest of the degrees of the three polynomials. Denote by a, b, and c the coefficients of x^n at $P(x)$, $Q(x)$, and $R(x)$, respectively (some of those coefficients might vanish).

The coefficients of x^n at $P(x) + Q(x) + R(x)$, as well as of x^{n^2} at $P(Q(x)) + Q(R(x)) + R(P(x))$, both vanish. Hence,

$$a + b + c = 0 \qquad \text{and} \qquad ab^n + bc^n + ca^n = 0. \tag{$*$}$$

Further we make use only of the two equalities in $(*)$.

The first equality yields that at least two numbers among a, b, and c are nonzero. If the third number (say, c) vanishes, then the second equality is violated. Therefore, all three polynomials have degree n, and we need to prove n is even.

Assume the contrary, for the sake of contradiction. Without loss of generality, the numbers a and b have the same sign. Changing the sign of all three numbers a, b, and c, if necessary, we achieve $a, b > 0$ (this change does not break $(*)$). Then we have $c = -(a + b) < 0$ and $0 < a, b < |c|$; hence bc^n and ca^n are negative, and therefore

$$|bc^n + ca^n| > |bc^n| = |c| \cdot b \cdot |c|^{n-1} > a \cdot b \cdot b^{n-1} = ab^n.$$

This contradicts the secund equality in $(*)$.

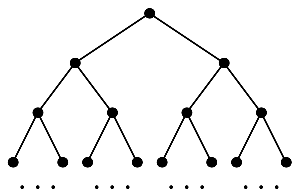
Thus, all three degrees are equal to an even number n.

Marking scheme

The points provided for differenmt parts are automatically additive!

Part 1: $\deg P = \deg Q = \deg R$.
A proof that all three degrees are equal — *2 points*
Only a proof that the two largest degrees are equal — *0 points*

Part 2; all thjjree degrees are even.
A proof that the largest degree d is even – *5 points*
A system $(*)$ is written down explicitly, with no subsequent essential advantage — *1 point instead of 5.*



№2. A ten-level 2-tree is drawn in the plane: a vertex A_1 is marked, it is connected by segments with two vertices B_1 and B_2, each of B_1 and B_2 is connected by segments with two of the four vertices C_1, C_2, C_3, C_4 (each C_i is connected with one B_j exactly); and so on, up to 512 vertices $J_1, \ldots, J_{512}$. Each of the vertices $J_1, \ldots, J_{512}$ is coloured blue or golden. Consider all permutations f of the vertices of this tree, such that (i) if X and Y are connected with a segment, then so are $f(X)$ and $f(Y)$, and (ii) if X is coloured, then $f(X)$ has the same colour. Find the maximum M such that there are at least M permutations with these properties, regardless of the colouring.

Solution. The answer is 2^{2^7}.

First we ned a suitable terminology. Similarly to 10-level 2-tree we can define a k-level 2-tree for $k \geq 1$. For convenience we suppose that all the segments between vertices are directed from a letter to the next one. The number of the letter marking a vertex we call the *level* of this vertex; thus A_1 is the only vertex of level 1, B_1 and B_2 belong to level 2 and so on). We will also call *descendants* of a vertex X all vertices which can be reached from X by directed segments.

Let T_1 and T_2 be two k-level 2-trees with coloured leaves. We call a bijection $f : T_1 \to T_2$ *isomorphism* when two conditions are satisfied: (i) if two vertices X and Y are connected by an edge in T_1, then $f(X)$ and $f(Y)$ are connected by an edge in T_2, and (ii) if X has some colour in T_1, then $f(X)$ has the same colour in T_2. When $T_1 = T_2$, we call f *automorphism* of the tree. By $\chi(k)$ we denote the minimal number of automorphism a k-level 2-tree with coloured leaves can have (the minimum is over all colourings). Our problem is to find $\chi(10)$.

We start with almost obvious

Lemma 1. Isomorphism of trees preserves the level of a vertex.

Proof. Isomorphism f cannot diminish the degree of a vertex. Indeed, neighbours of each vertex X become neighbours of $f(X)$, therefore the degree of $f(X)$ is not less than the degree of X. By pigeonhole principle it also means that the degree can not increase. It follows that the last level vertices go to the last level vertices. Therefore vertices of the previous level go to the same level, since they remain neighbours of the last-level vertices, and so on.

Now we are ready to solve the problem.

First proof of the lower bound, by induction.

Proposition 1. For each $k \geq 2$ we have $\chi(k) \geq (\chi(k-1))^2$.

Proof. In a k-level tree the descendants of B_1 (including B_1) form a $k-1$-level tree T_1. This graph has at least $\chi(k-1)$ different automorphisms. The same is true for tree T_2 formed by the descendants of B_2. Let g and h be automorphisms of T_1 and T_2 respectively. Now we can define mapping f of the whole tree applying g to descendans of B_1, h to descendants of B_2 and A to itself. Obviously f is an automorphism: for $X = A$ the condition holds since B_1 and B_2 were mapped to themselves (by Lemma 1), and for X in T_1 or T_2 because g and h are automorphisms. Thus for each pair (g, h) there is an automorphism f, different pairs produce different f, and the number of pairs is at least $(\chi(k-1))^2$.

Corollary. For $k \geq 3$ we have $\chi(k) \geq 2^{2^{k-3}}$.

Proof. This inequality is proved by induction, with Proposition 1 as induction step. It remains to check it for $k = 3$. If in a 3-level 2-tree at least one of the vertices B_1, B_2 has two descendants of the same colour, there is an automorphism exchanging these two vertices and preserving the rest. If each of B_1, B_2 has obe blue and one golden descendant, there is an automorphism exchanging B_1 and B_2 and preserving colours of their descendant. In both cases the number of automorphisms (including the identical one) is at least 2.

Second proof of the lower bound, without induction.

We already know that every 3-level 2-tree with (four) coloured leaves there are at least two colour-preservin automorphisms. Now every n-level tree, $n \geq 3$, has 2^{n-3} vertices of level $n-2$, and the descendants of each of these vertices form a 3-level tree. It is enough to consider automorphisms preserving vertices of level $n-3$ (and, a fortiori, of all lesser levels). Such an automorhism can act on the descendants of each of 2^{n-3} vertices of level $n-2$ in at least 2 ways. Thus there are at least $2^{2^{n-3}}$ such automorphisms. $\square$

It remains to construct for each $k \geq 3$ a colouring of k-level tree a colouring admitting exactly $2^{2^{k-3}}$ automorphisms. As it happens sometimes, we will prove somewhat more.

Proposition 2. For each $k \geqslant 3$ there are three colourings $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$ of leaves of k-level 2-tree such that the trees with these colourings are not isomorphic, and each of these colourings admits $2^{2^{k-3}}$ automorphisms exactly.

Proof. For $k = 3$ let C_1, C_2 be the descendants of B_1, and C_3, C_4 the descendants of B_2. The three colourings are the following: C_1, C_2, C_3 blue, C_4 golden; C_1, C_2, C_3 golden, C_4 blue; C_1, C_3 blue, C_2, C_4 golden. Obviously the trees with these colourings are not isomorphic and admit two automorphisms each.

The induction step. Let $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$ be the desired colourings of k-level tree. Consider the following colourings of the $(k+1)$-level tree:

- $\mathcal{M}_1$ for descendants of B_1 and $\mathcal{M}_2$ for descendants of B_2;
- $\mathcal{M}_2$ for descendants of B_1 and $\mathcal{M}_3$ for descendants of B_2;
- $\mathcal{M}_3$ for descendants of B_1 and $\mathcal{M}_1$ for descendants of B_2.

It is quite obvious that these three colourings are not isomorphic and have the desired number of automorphisms.

Comment to the example. Note that in fact we solved the following problem: find a colouring of $(n-2)$-level tree in 3 colours such that only identical automorphism preserves the colours. Indeed, there are three mutually non-isomorphic colourings of 3-level tree in 2 colours having only 2 automorphisms. We want the colouring of the descendants of each vertex of level $n-2$ to be one of these colourings. The correspondence between vertices of level $n-2$ and these three colouring must be the desired colouring of $n-2$-level tree admitting only identical automorphism.

Marking scheme

1. Answer: .... **1 point**

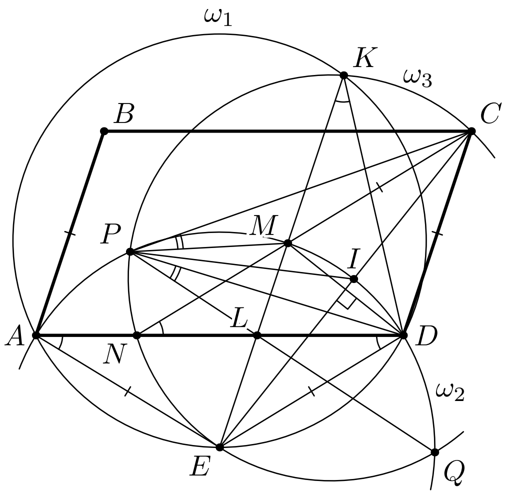
2. Lemma 1: .... **0 points**
(and points are not deducted if Lemma 1 is not proved)

3. Example and lower bound: .... **3 points each**
not additive with (1)

№3. In parallelogram $ABCD$ with acute angle A a point N is chosen on the segment AD, and a point M on the segment CN so that $AB = BM = CM$. Point K is the reflection of N in line MD. The line MK meets the segment AD at point L. Let P be the common point of the circumcircles of AMD and CNK such that A and P share the same side of the line MK. Prove that $\angle CPM = \angle DPL$.

Solution. Since $CM = AB = CD$, the triangle CMD is isosceles. Therefore, $\angle CDM + \angle DMK = \angle CMD + \angle DMN = 180°$, and hence $MK \parallel CD$.

Let E be the reflection of C in the line MD. Then both quadrilaterals $DCME$ and $ABME$ are rhombi with equal side lengths, as $ME \parallel CD \parallel AB$ and $ME = MC = CD = AB = BM$. Now, $MK \parallel CD$ implies that the point E lies on the line KL. Taking into account that $AE = DE$, we obtain $\angle DKE = \angle DKM = \angle DNM = \angle NDE = \angle NAE$. So the quadrilateral $AEDK$ is cyclic in some circle ω_1.



Let ω_2 and ω_3 denote the circumcircles of the triangles AMD and CNK, respectively (since $AE = DE = ME$, the point E is the center of ω_2). By symmetry in MD, the quadrilateral $CKNE$ is an isosceles trapezoid, so the point E lies on ω_3. Let ω_2 and ω_3 meet again at Q. The point $L = AD \cap KE$ is the radical center of the circles ω_1, ω_2, and ω_3, so L lies on line PQ.

Let the ray EC meet ω_2 at I. Then the arcs IM and ID in circle ω_2 are congruent, so that I lies on the internal angle bisector of $\angle DPM$. But the point I lies also on the internal angle bisector of $\angle CPQ$, since $\angle QPI = \angle QEI/2 = \angle QEC/2 = \angle QPC/2$. Therefore, the lines PM and PD are symmetric to each other with respect to the internal angle bisector of $\angle QPC$, which yields the desired equality $\angle CPM = \angle DPL$.

Remark 1. The points M and D are isogonally conjugate with respect to the triangle CPQ, while I is the incenter of that triangle.

Remark 2. The point M is the incenter of the triangle AKD.

Marking scheme

An unfinished analytical solution (by means of Cartesian coordinates, complex numbers, vectors, trigonometric formulas, etc.): .... **0 points**

Partial score

The points listed below are to be added to each other.

Let Q denote the second meeting point of the circles (AMD) and (CNK).

1. A proof that $MK \parallel CD$: .... **1 point**
2. A proof that the quadrilateral $AEDK$ is cyclic: .... **2 points**
3. A proof that L lies on the line PQ: .... **1 point**
4. A reduction of the problem statement to the fact that L lies on PQ .... **3 points**