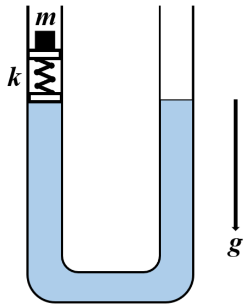
THEORETICAL COMPETITION
January 8, 2021

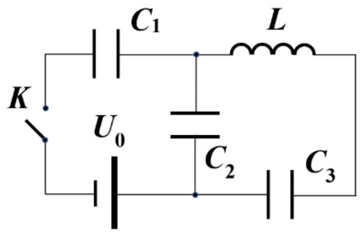
Problem 1 (10.0 points)
This problem consists of three independent parts.

Problem 1.1 (4.0 points)
Water of mass density $\rho = 1.00 g/sm^3$ is poured into a vertical U-shaped tube of small constant cross-section $s = 8.00 sm^2$, such that the total length of water in both legs is $l = 50.0 sm$. Two pistons are placed into one leg of the tube with the spring of stiffness $k = 1.00 N/m$ in between. The pistons are watertight and can slide along the tube without friction. At the initial moment, a weight of mass $m = 10.0 g$ is placed on the upper piston. Determine the possible frequencies of small harmonic vibrations of the system near a new equilibrium position. The mass of the pistons and the spring can be neglected, the acceleration of gravity is equal to $g = 9.80 m/s^2$. Consider water as an ideal incompressible liquid.



Problem 1.2 (3.0 points)
Some insects, such as water striders, are capable of moving freely on the water surface, because their legs are densely covered with non-wetting hairs. To understand why this turns possible, consider the following model problem. A square plate with the side $a = 10.0 sm$ and thickness $h = 2.00 mm$ is carefully laid on the water surface. The density of the plate material is equal to $\rho = 1.10 g/sm^3$, the density of water is $\rho_0 = 1.00 g/sm^3$, and its surface tension constitutes $\sigma = 7.30 \cdot 10^{-2} N/m$. Find the maximum mass m of a weight that can be put on the plate so that it still does not sink. Assume that the weight does not deform the plate, the acceleration of free fall is $g = 9.80 m/s^2$.

Problem 1.3 (3.0 points)
The electrical circuit shown schematically in the figure consists of three capacitors with capacities C_1, C_2 and C_3, a coil of inductance L and a source of constant voltage U_0. At the initial moment of time, the capacitors are not charged, and the current in the coil is zero. The switch K is shorted. Find the maximum current $I_{\max}$ through the coil and determine the minimum voltage $U_{\min}$ across the capacitor C_2. Assume that the resistance of the connecting wires is rather small.



Problem 2. Thermodynamics of one-component plasma (10.0 points)

Plasma, considered a fourth state of matter, is an ionized gas containing electrons, ions and neutral particles. In a plasma, particle concentrations and temperatures vary over a very wide range, so that a great variety of physical effects can play an essential role. Therefore, at present, a great deal of plasma models have been developed, and this problem deals with one of them, which is called a one-component plasma model. Namely, let us consider a fully ionized plasma with no neutral particles present, which consists of positively charged deuterium nuclei moving against a neutralizing uniformly charged background formed by electrons. This model is a very good approximation for ultrahigh-pressure plasmas occurring at the center of white dwarfs and giant planets like Jupiter. Let the charge and the mass of deuterium nuclei be $e = 1.602 \cdot 10^{-19} Cl$ and $m = 3.44 \cdot 10^{-24} g$ respectively, and their concentration be $n = 1,62 \cdot 10^{27} sm^{-3}$ at temperature $T = 1.76 \cdot 10^{4} K$. Under these conditions, an essential role is played by the interaction between deuterium nuclei, which are located at the sites of a cubic lattice, whose two-dimensional projection is shown in Figure 2.1. Plasma must preserve its neutrality, therefore, each cube with a nucleus located in its center is neutral and referred to as a unit cell. The field produced by each cubic cell is rather complicated, and instead the system is formally reduced to spherical cells, whose two-dimensional projection is depicted in Figure 2.2. The justification of such a replacement is not obvious and depends on the type of problems under investigation.

In numerical calculations, consider the following known: Boltzmann's constant $k_B = 1.38 \cdot 10^{-23} J/K$, the vacuum permittivity $\varepsilon_0 = 8,85 \cdot 10^{-12} F/m$.

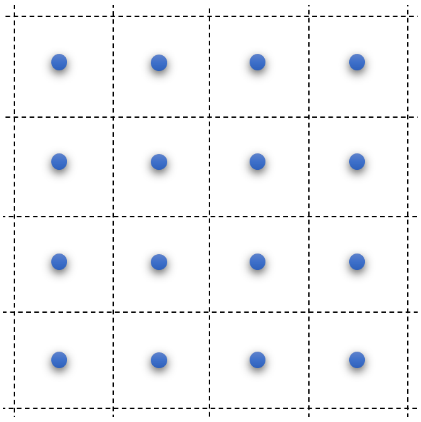


Figure 2.1. Two-dimensional projection of a one-component plasma model with cubic cells.

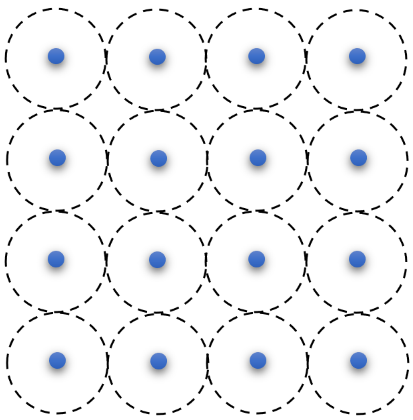


Figure 2.2. Two-dimensional projection of a one-component plasma model with spherical cells.

2.1 Calculate the smallest distance a between neighboring nuclei.

2.2 Show that interaction energy between nuclei plays a significant role under the given conditions. To do this, estimate the ratio Γ of the interaction energy of neighboring nuclei to their thermal energy. Neglect the presence of a neutralizing background.

2.3 Calculate the bulk charge density ρ of a spherical cell in a one-component plasma model.

2.4 Calculate the potential difference between two points of a spherical cell located at distances $a/2$ and $a/4$ respectively.

2.5 Calculate the frequency of small oscillations ω_p of a nucleus near the equilibrium position in a spherical cell.

2.6 At the given plasma temperature, estimate the rms amplitude A of oscillations of nuclei near their equilibrium position.

2.7 The internal energy U of a one-component plasma, containing N spherical cells in the volume V, has the form

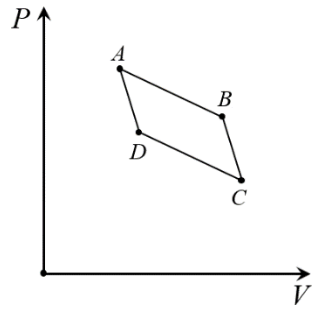
$$U = \alpha_1 N + \alpha_2 \frac{N^{4/3}}{V^{1/3}}.$$

Determine constants α_1 and α_2.

The plasma state of matter is a promising working body for the controlled nuclear fusion. The main problem in the implementation of nuclear fusion is to overcome the so-called Coulomb barrier, which is the Coulomb repulsion between positively charged nuclei. Note that the presence of the neutralizing background of nuclei results in lowering of the Coulomb barrier, since the repulsive force between nuclei decreases. Consider the process of two fusing cells, which occurs as follows. Two cells fuse into one spherical cell with the same bulk density of the neutralizing background, and a new nucleus appears in its center, formed by the fusion of two initial nuclei.

2.8 Calculate the Coulomb barrier lowering for the fusion of two deuterium nuclei cells under given conditions.

The expression for the internal energy of a one-component plasma in the vell model in 2.7 above is interesting in that it explicitly depends on the volume, which is characteristic for nonideal systems. Let a thermodynamic state of the system, whose composition remains unchanged, be depicted by a dot on the pressure (P) – volume (V) diagram. In this diagram consider a process consisting of two isotherms AB and CD, as well as two adiabats BC and AD. Variations in volumes, temperatures and pressures in this process may be considered so small that the quadrilateral $ABCD$ can be assumed a parallelogram.



2.9 Using the above cycle, express the derivative $(\partial U / \partial V)_T$ of the internal energy with respect to volume at a fixed temperature in terms of the derivative $(\partial P / \partial T)_V$ of the pressure with respect to temperature at a fixed volume as well as the temperature T and pressure P of the system.

2.10 The pressure P of a one-component plasma, containing N spherical cells in the volume V, has the form

$$P = \beta_1 \frac{N}{V} + \beta_2 \left(\frac{N}{V}\right)^{\beta_3}.$$

Determine constants β_1, β_2 and β_3. Calculate the numerical value of the pressure for the plasma parameters given in the problem statement.

Problem 3. Optical waveguide (10.0 points)

At present, various waveguides are widely used to transmit energy and information. The propagation of electromagnetic waves in waveguides differs significantly from the propagation of waves in free space, and in this problem you are asked to describe the propagation of electromagnetic waves in the simplest plane waveguide.

Description of waves

A plane monochromatic electromagnetic wave propagating along the axis Ox is described by the formula

$$\vec{E}(t,x)=\vec{E}_0\cos(\omega t-kx+\varphi). \qquad (1)$$

Here $\vec{E},\vec{E}_0$ denote the electric field strength of the wave and its amplitude, respectively, the magnitude of k is called the wavenumber, ω refers to the circular wave frequency, φ signifies the initial phase, and the expression in the cosine is called the wave phase.

3.1 Express the wavenumber k in terms of the wavelength λ and the period of oscillation T in terms of the angular frequency ω.

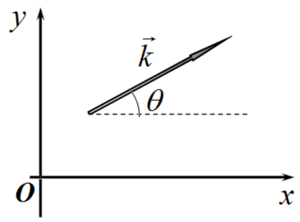
3.2 Express the velocity c of wave propagation in terms of k and ω.

In a more general case, a monochromatic plane wave is described by the function

$$\vec{E}(t,\vec{r})=\vec{E}_0\cos\left(\omega t-\vec{k}\cdot\vec{r}+\varphi\right). \qquad (2)$$

In this expression, $\vec{r}$ stands for the radius vector of an arbitrary point in space, $\vec{k}$ designates the wave vector equal in magnitude to the wavenumber and pointing to the direction of wave propagation.

Let the wave vector $\vec{k}$ lie in the plane Oxy and be directed at an angle θ to the axis Ox, as shown in the figure to the right.

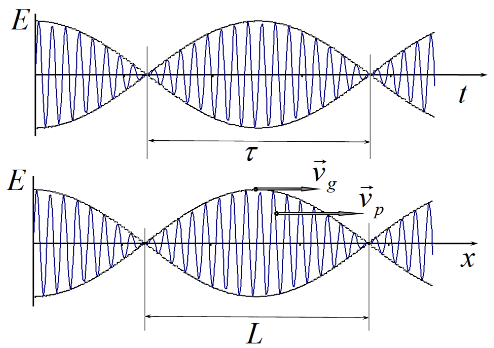


3.3 Draw schematically a family of equidistant wave surfaces, which are surfaces of equal wave phases, for a plane wave described by function (2).

3.4 Write down explicitly the expression for the electric field strength of wave (2) as a function of coordinates $\vec{E}(t,x,y)$.

An ideal monochromatic wave is infinite in time and space, therefore, it cannot carry information. To transmit information, it is necessary to use either separate pulses (restricted in time) or variate the wave amplitude (wave modulation). In these cases, the wave ceases to be monochromatic, and can be represented as a sum (superposition) of monochromatic waves.

Consider a wave that is the sum of two waves propagating along the axis Ox: the first one having the angular frequency ω_0 and wavenumber k_0; the second wave frequency is $\omega_0+\Delta\omega$, $\Delta\omega \ll \omega_0$, with the wavenumber $k_0+\Delta k, \Delta k \ll k_0$. Note that, in the general case, the wavenumber is a certain function of frequency $k(\omega)$.



3.5 Show that the sum of these two monochromatic waves is a modulated wave consisting of separate wave packets. Write down the formula describing the slow variation of the amplitude $A_0(x,t)$ of the resulting wave in space and time (it is called an envelope).

3.6 Determine the time duration of an individual wave packet τ. Write down the relationship between the duration τ and the frequency difference $\Delta\nu=\Delta\omega/2\pi$.

3.7 Determine the spatial length of the wave packet L.

It turns out that the speed of the wave surface of constant phase v_p, which is called the phase velocity, differs from the speed of the wave packet v_g, which is called the group velocity. The speed of the envelope maximum can be considered a group velocity.

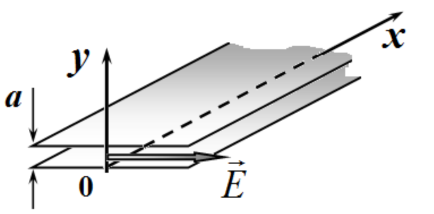
3.8 Find the phase velocity v_p of the considered modulated wave and express it in terms of $\omega, k, \Delta\omega, \Delta k$.

3.9 Find the group velocity v_g of the considered modulated wave and express it in terms of $\omega, k, \Delta\omega, \Delta k$.

3.10 Establish a relationship between phase v_p and group v_g velocities for electromagnetic waves in a vacuum.

Plane waveguide

In this part, consider the propagation of electromagnetic waves in a plane waveguide. The waveguide is formed by two infinite parallel conductive plates located at a distance a from each other. Assume a vacuum in between the plates.



Under investigation are electromagnetic waves, whose electric field strength vectors are directed parallel to the plates (they are called TE waves). Let us introduce a coordinate system, whose Ox axis lies in one of the plates, and whose Oy axis is directed perpendicular to the plates.

A wave propagating along the axis Ox in between the plates is described by the function

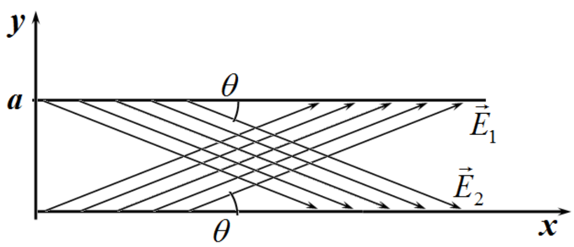
$$E(t,x,y) = E_0 \cos(\omega t - k_x x)\sin(k_y y), \qquad (3)$$

where ω is the known circular frequency of the wave. For this wave to propagate in the waveguide without energy losses, the electric field strength on the plates must be zero.

3.11 Find the values k_y at which the wave can propagate in the waveguide without energy losses.

A set of possible values k_y is discrete and characterized by some integer m. Waves corresponding to different values of this number are called modes (types of possible waves).

3.12 Show that the wave described by function (3) can be represented as a superposition of two plane waves $E_1(t,x,y)$ and $E_2(t,x,y)$ with the wave numbers k_0, propagating symmetrically at angles $\pm\theta$ to the plates, see figure below.



3.13 Express the values k_x, k_y in terms of the wavenumber k_0 and angle θ.

3.14 Determine the possible angles θ_m at which the wave can propagate in the waveguide without energy losses. Express the values of these angles in terms of the distance a between the plates and the wavelength λ in vacuum.

3.15 Determine the phase velocities v_p of the waves of each mode. Express these velocities in terms of the wave frequency ω and the speed of light c in a vacuum.

Short pulses with a carrier frequency ω_0 are supplied to the waveguide input. Since these pulses have a finite time duration τ, they cannot be treated as a monochromatic wave, but instead contain a set of monochromatic components in a certain frequency range $\Delta\omega \ll \omega_0$. These input pulses form a set of pulses in each of the possible waveguide modes.

3.16 Determine the speed of the pulse propagation in the mode number m.

3.17 At what minimum distance X from the waveguide input the number of pulses is to be doubled if $a/\lambda = 1.2$. Express your answer in terms of the speed of light c and the pulse duration τ.

To avoid the appearance of "extra" pulses at the information transmission process, waveguides are used to operate in a single-mode regime, in which only one mode can propagate.

3.18 Find ratios a/λ at which only one mode can propagate in the waveguide.

Mathematical hints for the theoretical problems

The following formulas may be useful:

$\int x^n dx = \frac{x^{n+1}}{n+1}$, where n is an integer;

$(1+x)^\gamma \approx 1 + \gamma x + \frac{\gamma(\gamma-1)}{2}x^2$, *for* $x \ll 1$ *and any* γ.