XIV International Zhautykov Olympiad in Mathematics
Almaty, 2020

January 10, 9.00-13.30
First day
(Each problem is worth 7 points)

1. A positive integer n does not divide $2^a 3^b + 1$ for any positive integers a and b. Prove that n does not divide $2^c + 3^d$ for any positive integers c and d.

Solution. Assume the contrary: n divides $2^c + 3^d$. Clearly n is not divisible by 3; therefore n divides $3^k - 1$ for some k. Choosing s so that $ks > d$ we see that n divides $3^{ks-d}(2^c + 3^d) = 2^c 3^{ks-d} + 3^{ks}$. Then n also divides $2^c 3^{ks-d} + 1 = 2^c 3^{ks-d} + 3^{ks} - (3^{ks} - 1)$, a contradiction.

2. In a set of 20 elements there are $2k+1$ different subsets of 7 elements such that each of these subsets intersects exactly k other subsets. Find the maximum k for which this is possible.

The answer is $k = 2$.

Solution. Let M be the set of residues mod20. An example is given by the sets $A_i = \{4i+1, 4i+2, 4i+3, 4i+4, 4i+5, 4i+6, 4i+7\} \subset M$, $i = 0, 1, 2, 3, 4$.

Let $k \geq 2$. Obviously among any three 7-element subsets there are two intersecting subsets.

Let A be any of the $2k+1$ subsets. It intersects k other subsets $B_1, \ldots, B_k$. The remaining subsets $C_1, \ldots, C_k$ do not intersect A and are therefore pairwise intersecting. Since each C_i intersects k other subsets, it intersects exactly one B_j. This B_j can not be the same for all C_i because B_j can not intersect $k+1$ subsets.

Thus there are two different C_i intersecting different B_j; let C_1 intersect B_1 and C_2 intersect B_2. All the subsets that do not intersect C_1 must intersect each other; there is A among them, therefore they are A and all B_i, $i \neq 1$. Hence every B_j and B_j, $i \neq 1$, $j \neq 1$, intersect. Applying the same argument to C_2 we see that any B_i and B_j, $i \neq 2$, $j \neq 2$, intersect. We see that the family A, $B_1, \ldots, B_k$ contains only one pair, B_1 and B_2, of non-untersecting subsets, while B_1 intersects C_1 and B_2 intersects C_2. For each i this list contains k subsets intersecting B_i. It follows that no C_i with $i > 2$ intersects any B_j, that is, there are no such C_i, and $k \leq 2$.

3. A convex hexagon $ABCDEF$ is inscribed in a circle. Prove the inequality

$$AC \cdot BD \cdot CE \cdot DF \cdot AE \cdot BF \geq 27 AB \cdot BC \cdot CD \cdot DE \cdot EF \cdot FA.$$

Solution. Let

$$d_1 = AB \cdot BC \cdot CD \cdot DE \cdot EF \cdot FA, d_2 = AC \cdot BD \cdot CE \cdot DF \cdot AE \cdot BF, d_3 = AD \cdot BE \cdot CF.$$

Applying Ptolemy's theorem to quadrilaterals $ABCD$, $BCDE$, $CDEF$, $DEFA$, $EFAB$, $FABC$, we obtain six equations $AC \cdot BD - AB \cdot CD = BC \cdot AD, \ldots, FB \cdot AC - FA \cdot BC = AB \cdot FC$. Putting these equations in the well-known inequality

$$\sqrt[6]{(a_1 - b_1)(a_2 - b_2) \cdot \ldots \cdot (a_6 - b_6)} \leq \sqrt[6]{a_1 a_2 \ldots a_6} - \sqrt[6]{b_1 b_2 \ldots b_6} \quad (a_i \geq b_i > 0, i = 1, \ldots, 6),$$

we get

$$\sqrt[3]{d_3}\sqrt[6]{d_1} \leq \sqrt[3]{d_2} - \sqrt[3]{d_1}. \quad (1)$$

Applying Ptolemy's theorem to quadrilaterals $ACDF$, $ABDE$ и $BCEF$, we obtain three equations $AD \cdot CF = AC \cdot DF + AF \cdot CD$, $AD \cdot BE = BD \cdot AE + AB \cdot DE$, $BE \cdot CF = BF \cdot CE + BC \cdot EF$. Putting these equations in the well-known inequality

$$\sqrt[3]{(a_1 + b_1)(a_2 + b_2)(a_3 + b_3)} \geq \sqrt[3]{a_1 a_2 a_3} + \sqrt[3]{b_1 b_2 b_3} \quad (a_i > 0, b_i > 0, i = 1, 2, 3),$$

we get

$$\sqrt[3]{d_3^2} \geq \sqrt[3]{d_2} + \sqrt[3]{d_1}. \quad (2)$$

It follows from (1) and (2) that $(\sqrt[3]{d_2} - \sqrt[3]{d_1})^2 \geq \sqrt[3]{d_1}(\sqrt[3]{d_2} + \sqrt[3]{d_1})$, that is, $\sqrt[3]{d_2} \geq 3\sqrt[3]{d_1}$ and $d_2 \geq 27 d_1$, q.e.d.