XV International Zhautykov Olympiad in Mathematics

Second day. Solutions

№4. An isosceles triangle ABC with $AC = BC$ is given. Point D is chosen on the side AC. The circle S_1 of radius R with the center O_1 touches the segment AD and the extensions of BA and BD over the points A and D, respectively. The circle S_2 of radius $2R$ with the center O_2 touches the segment DC and the extensions of BD and BC over the points D and C, respectively. Let the tangent to the circumcircle of the triangle BO_1O_2 at the point O_2 intersect the line BA at point F. Prove that $O_1F = O_1O_2$.

Solution By condition, in the triangle ABC we have $\angle A = \angle B$. It is evident that $\angle O_1BO_2 = \angle B/2$. Let ℓ be the straight line passing through O_2 parallel to AC. By the problem condition ℓ touches S_1 (say, at a point N). Let also K be the tangency point of S_1 and BA. Then the clockwise rotation about the point O_1 through the angle NO_1K transposes ℓ to BA and thus transposes the point O_2 to some point $O \in BA$. Hence $O_1O = O_1O_2$ and $\angle OO_1O_2 = \angle NO_1K = 180° - \angle A = 180° - \angle B$, so $\angle O_1O_2O = \angle B/2 = \angle O_1BO_2$. The latter does mean that the line O_2O is the tangent to the circumcircle of $\triangle BO_1O_2$. Hence $F = O$, and $O_1F = O_1O_2$, as was to be proved.

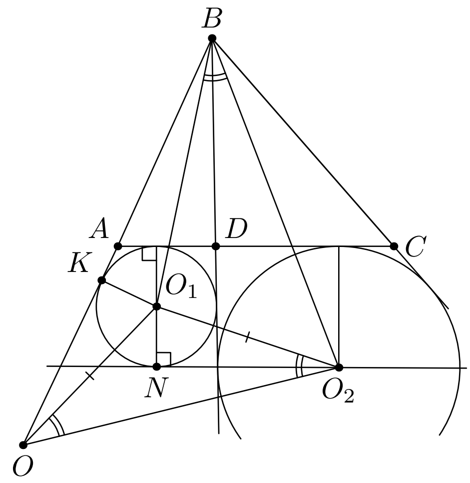

Рис. 1

№5. Let $n > 1$ be a positive integer. A function $f : I \to \mathbb{Z}$ is given, where I is the set of all integers coprime with n. ($\mathbb{Z}$ is the set of integers). A positive integer k is called a *period* of the function f if $f(a) = f(b)$ for all $a, b \in I$ such that $a \equiv b \pmod{k}$. It is known that n is a period of f. Prove that the minimal period of the function f divides all its periods.

Example. For $n = 6$, the function f with period 6 is defined entirely by its values $f(1)$ and $f(5)$. If $f(1) = f(5)$, then the function has minimal period $P_{\min} = 1$, and if $f(1) \neq f(5)$, then $P_{\min} = 3$.

№6. On a polynomial of degree three it is allowed to perform the following two operations arbitrarily many times:

(i) reverse the order of its coefficients including zeroes (for instance, from the polynomial $x^3 - 2x^2 - 3$ we can obtain $-3x^3 - 2x + 1$);

(ii) change polynomial $P(x)$ to the polynomial $P(x+1)$.

Is it possible to obtain the polynomial $x^3 - 3x^2 + 3x - 3$ from the polynomial $x^3 - 2$?

The **answer** is no.

Solution I. The original polynomial $x^3 - 2$ has a unique real root. The two transformations clearly preserve this property. If α is the only real root of $P(x)$, then the first operation produces a polynomial with root $\frac{1}{\alpha}$, and the second operation gives a polynomial with root $\alpha - 1$. Since the root of the original polynomial is $\sqrt[3]{2}$, and thar of the resulting polynomial is $1 + \sqrt[3]{2}$, the problem is reduced to the question whether it is possible to obtain the latter number from the former by operations $x \mapsto \frac{1}{x}$ and $x \mapsto x - 1$. Let us apply one more operation $x \mapsto x - 1$ (so as to transform $\sqrt[3]{2}$ to itself) and reverse all the operations. It appears then that the number $\sqrt[3]{2}$ is transformed to itself by several operations of the form $x \mapsto \frac{1}{x}$ and $x \mapsto x + 1$. It is easy to see that the composition of any number of such operations is a fractional-linear function $x \mapsto \frac{ax+b}{cx+d}$, where a, b, c, d are non-negative integers and $ad - bc = 1$. Each operation $x \mapsto x + 1$ increases $a + b + c + d$, and, since we started with this operation, the resulting function is not identical. Thus $\sqrt[3]{2}$ is transformed to itself by some such composition. This means however that $\sqrt[3]{2}$ is a root of non-zero polynomial $x(cx+d) - ax - b$ with integral coefficients and degree at most 2, which is impossible.

Solution II. The original polynomial has one real and two conjugate complex roots. We have seen above that under the two operations these roots are subject to transforms $x \mapsto \frac{1}{x}$ and $x \mapsto x - 1$. Note that both imaginary roots of the original polynomial have negative real part. It is easy to check that this property is preserved under the two operations. However the real parts of all the roots of the desired polynomial are positive, a contradiction.

Solution III. For a polynomial $P(x) = ax^3 + bx^2 + cx + d$ we define $\Delta(P) = 3ad - bc$. The first operation transforms $P(x)$ to $dx^3 + cx^2 + bx + a$ and does not change Δ. The second operation transforms $P(x)$ to $Q(x) = ax^3 + (b + 3a)x^2 + (c + 3a + 2b)x + (d + a + b + c)$, for which $\Delta(Q) = 3(d + a + b + c)a - (b + 3a)(c + 3a + 2b) = \Delta(P) - (2b^2 + 6ab + 6a^2) < \Delta(P)$. Thus the permitted operation can not increase Δ. On the other hand, for the original polynomial $\Delta(P) = -6$, and for the resulting polynomial it must be 0.