XV International Zhautykov Olympiad in Mathematics
Almaty, 2019
January 11, 9.00-13.30

№1. Prove that there are at least 100! ways to partition the number 100! into summands from the set $\{1!, 2!, 3!, \ldots, 99!\}$. (Partitions differing in the order of summands are considered the same; any summand can be taken multiple times. We remind that $n! = 1 \cdot 2 \cdot \ldots \cdot n$.)

Solution. Let us prove by induction on $n \geqslant 4$ that there are at least $n!$ ways to partition the number $n!$ into summands from $\{1!, 2!, \ldots, (n-1)!\}$.

For $n = 4$, if we use only the summands $1!, 2!$ there are 13 ways to partition 4! as 2! can be used from 0 to 12 times. If 3! is used 1 time, then $4! - 3! = 18$ can be partitioned using $1!, 2!$ in 10 ways. We get at least one more partition if we use 3! two times. So, there are at least 24 such partitions as needed.

Suppose now the statement holds for n and let us prove it for $n+1$. To partition $(n+1)!$, the summand $n!$ can be used i times for $0 \leqslant i \leqslant n$. By the hypothesis, for every such i, the remaining number $(n+1)! - i \cdot n! = (n+1-i) \cdot n!$ can be partitioned into the summands $\{1!, \ldots, (n-1)!\}$ in at least $n!$ ways as follows. For any partition of $n!$ take each summand appearing say k times and write it $(n+1-i)k$ times. Hence we obtain at least $(n+1) \cdot n! = (n+1)!$ ways to partition the number $(n+1)!$ as desired. The original problem follows for $n = 100$ then.

№2. Find the largest real C such that for all pairwise distinct positive real $a_1, a_2, \ldots, a_{2019}$ the following inequality holds

$$\frac{a_1}{|a_2 - a_3|} + \frac{a_2}{|a_3 - a_4|} + \ldots + \frac{a_{2018}}{|a_{2019} - a_1|} + \frac{a_{2019}}{|a_1 - a_2|} > C.$$

2. **The answer** is 1010.

Solution. Without loss of generality we assume that $\min(a_1, a_2, \ldots, a_{2019}) = a_1$. Note that if a, b, c $(b \neq c)$ are positive, then $\frac{a}{|b-c|} > \min(\frac{a}{b}, \frac{a}{c})$. Hence

$$S = \frac{a_1}{|a_2 - a_3|} + \cdots + \frac{a_{2019}}{|a_1 - a_2|} > 0 + \min\left(\frac{a_2}{a_3}, \frac{a_2}{a_4}\right) + \cdots + \min\left(\frac{a_{2017}}{a_{2018}}, \frac{a_{2017}}{a_{2019}}\right) + \frac{a_{2018}}{a_{2019}} + \frac{a_{2019}}{a_2} = T.$$

Take $i_0 = 2$ and for each $\ell \geqslant 0$ let $i_{\ell+1} = i_\ell + 1$ if $a_{i_\ell+1} > a_{i_\ell+2}$ and $i_{\ell+1} = i_\ell + 2$ otherwise. There is an integral k such that $i_k < 2018$ and $i_{k+1} \geqslant 2018$. Then

$$T \geqslant \frac{a_2}{a_{i_1}} + \frac{a_{i_1}}{a_{i_2}} + \cdots + \frac{a_{i_k}}{a_{i_{k+1}}} + \frac{a_{2018}}{a_{2019}} + \frac{a_{2019}}{a_2} = A. \tag{1}$$

We have $1 \leqslant i_{\ell+1} - i_\ell \leqslant 2$, therefore $i_{k+1} \in \{2018, 2019\}$.

Since

$$2018 \leqslant i_{k+1} = i_0 + (i_1 - i_0) + \cdots + (i_{k+1} - i_k) \leqslant 2(k+2), \tag{2}$$

it follows that $k \geqslant 1007$. Consider two cases.

(i) $k = 1007$. Then in the inequality (2) we have equalities everywhere, in particular $i_{k+1} = 2018$. Applying AM–GM inequality for $k+3$ numbers to (1) we obtain $A \geqslant k + 3 \geqslant 1010$.

(ii) $k \geqslant 1008$. If $i_{k+1} = 2018$ then we get $A \geqslant k + 3 \geqslant 1011$ by the same argument as in the case (i). If $i_{k+1} = 2019$ then applying AM–GM inequality to $k+2$ summands in (1) (that is, to all the summands except $\frac{a_{2018}}{a_{2019}}$) we get $A \geqslant k + 2 \geqslant 1010$.

So we have $S > T \geqslant A \geqslant 1010$. For $a_1 = 1 + \varepsilon$, $a_2 = \varepsilon$, $a_3 = 1 + 2\varepsilon$, $a_4 = 2\varepsilon$, $\ldots$, $a_{2016} = 1008\varepsilon$, $a_{2017} = 1 + 1009\varepsilon$, $a_{2018} = \varepsilon^2$, $a_{2019} = 1$ we obtain $S = 1009 + 1008\varepsilon + \frac{1008\varepsilon}{1+1009\varepsilon-\varepsilon^2} + \frac{1+1009\varepsilon}{1-\varepsilon^2}$. Then $\lim\limits_{\varepsilon \to 0} S = 1010$, which means that the constant 1010 cannot be increased.

№3. The extension of median CM of the triangle ABC intersects its circumcircle ω at N. Let P and Q be the points on the rays CA and CB respectively such that $PM \parallel BN$ and $QM \parallel AN$. Let X and Y be the points on the segments PM and QM respectively such that PY and QX are tangent to ω. The segments PY and QX intersect at Z. Prove that the quadrilateral $MXZY$ is circumscribed.

Solution.

Lemma. The points K and L lie on the sides BC and AC of a triangle ABC. The segments AK and BL intersect at D. Then the quadrilateral $CKDL$ is circumscribed if and only if $AC - BC = AD - BD$.

Proof. Let $CKDL$ be circumscribed and its incircle touches LC, CK, KD, DL at X, Y, Z, T respectively (see Fig. 1). Then

$$AC - BC = AX - BY = AZ - BT = AD - BD.$$

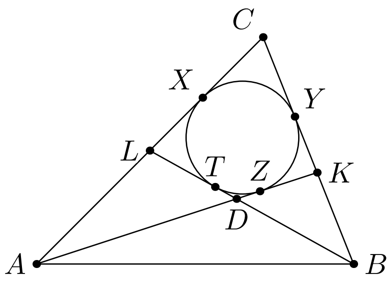

Рис. 1

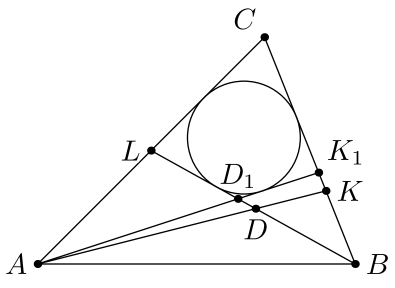

Рис. 2

Now suppose that $AC - BC = AD - BD$. Let the tangent to the incircle of BLC different from AC meets the segments BL and BC at D_1 and K_1 respectively. If $K = K_1$ then the lemma is proved. Otherwise $AD_1 - BD_1 = AC - BC = AD - BD$ or $AD_1 - BD_1 = AD - BD$. In the case when D lies on the segment BD_1 (see Fig. 2) we have

$$AD_1 - BD_1 = AD - BD \Rightarrow AD_1 - AD = BD_1 - BD \Rightarrow AD_1 - AD = DD_1.$$

But the last equation contradicts the triangle inequality, since $AD_1 - AD < DD_1$. The case when D is outside the segment BD_1 is similar.

Back to the solution of the problem, let PY and QX touch ω at Y_1 and X_1 respectively. Since $ACBN$ is cyclic and $PM \parallel BN$ we have $\angle ACN = \angle ABN = \angle AMP$, i. e. the circumcircle of $\triangle AMC$ is tangent to the line PM. Thus $PM^2 = PA \cdot PC$. But $PA \cdot PC = PY_1^2$, and therefore $PM = PY_1$. In the same way we have $QM = QX_1$. Obviously $ZX_1 = ZY_1$. It remains to note that the desired result follows from the Lemma because

$$PM - QM = PY_1 - QX_1 = (PZ + ZY_1) - (QZ + ZX_1) = PZ - QZ \quad \Rightarrow \quad PM - QM = PZ - QZ.$$

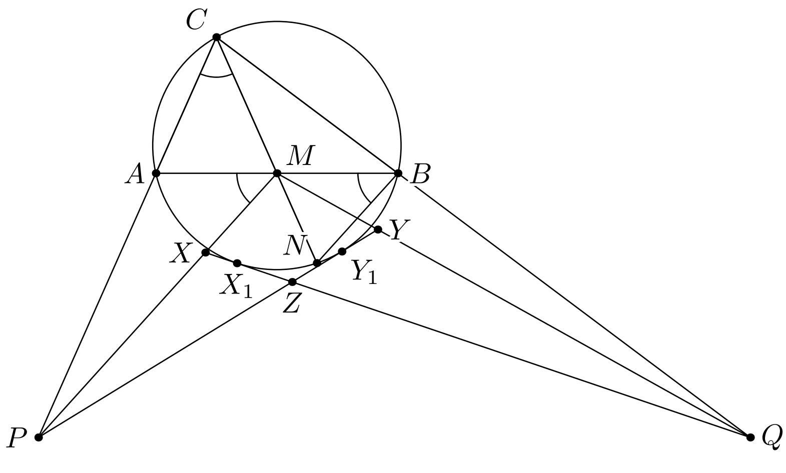

Рис. 3

Note. This solution does not use the comdition that M is the midpoint of AB.