Problem 4.

The answer is no.

Solution. Note that for each polynomial $P(x)$ with integral coefficients the integers a, b, c, d such that $P(1+\sqrt{3})=a+b\sqrt{3}$ and $P(3+\sqrt{5})=c+d\sqrt{5}$ are uniquely defined. We call a polynomial *regular* if $a-c$ and $b-d$ are even. If P, Q are regular and k is an integer, then $P+Q$ and kP are obviously regular. Let us prove that $R=PQ$ is also regular. Indeed, if $P(1+\sqrt{3})=a+b\sqrt{3}$, $P(3+\sqrt{5})=c+d\sqrt{5}$, $Q(1+\sqrt{3})=a'+b'\sqrt{3}$, $Q(3+\sqrt{5})=c'+d'\sqrt{5}$, then

$R(1+\sqrt{3})=(a+b\sqrt{3})(a'+b'\sqrt{3})=(aa'+3bb')+(ab'+ba')\sqrt{3}$,

$R(3+\sqrt{5})=(c+d\sqrt{5})(c'+d'\sqrt{5})=(cc'+5dd')+(cd'+dc')\sqrt{5}$.

Clearly if $a\equiv c \pmod 2$, $b\equiv d \pmod 2$, $a'\equiv c' \pmod 2$, $b'\equiv d' \pmod 2$, then $aa'+3bb'\equiv cc'+5dd' \pmod 2$ и $ab'+ba'\equiv cd'+dc' \pmod 2$.

Now the polynomial $P(x)=x$ is regular. It follows that so are all the polynomials with integral coefficients, therefore, the desired polynomial does not exist.

Solution of problem 5.

Note that it is enough to prove that $f(a,b,c) = f(b,a,c) = f(a,c,b)$. First, let us consider the following interpretation of our problem:

For every 6-tuple $(X_1, X_2, X_3, Y_1, Y_2, Y_3)$ satisfying conditions of the problem, we construct three sequences

$$A = (a_1, \ldots, a_{2014}),\ B = (b_1, \ldots, b_{2014}),\ C = (c_1, \ldots, c_{2014})$$

as follows:

for $i = 1,\ldots,2014$

$$a_i = \begin{cases} 2, \text{ if } i \in Y_1 \\ 1, \text{ if } i \in X_1 \setminus Y_1, \\ 0, \text{ otherwise.} \end{cases}$$

Similarly, we define sequences B, C. Conditions (i), (ii), (iii) imply the following conditions for sequences (A, B, C):

(P1) number of nonzero elements in A is a; number of nonzero elements in B is b; number of nonzero elements in C is c;

(P2) if $a_i = 2$ for some i, then $b_i = c_i = 0$; if $b_i = 2$, then $c_i = 0$.

Clearly, for every sequences (A, B, C) satisfying (P1), (P2) we may construct a sequence $(X_1, X_2, X_3, Y_1, Y_2, Y_3)$ that satisfies (i), (ii), (iii) of the problem.

So, $f(a,b,c)$ is a number of sequences (A, B, C) satisfying (P1), (P2).

Let us first prove that $f(a,b,c) = f(b,a,c)$. We establish the bijection Φ_1 between triples corresponding to the order (a, b, c) and (b, a, c) as follows

$$\Phi_1((A, B, C)) = (A', B', C),$$

where $A' = (a'_1, \ldots, a'_{2014})$, $B' = (b'_1, \ldots, b'_{2014})$ and for all $i = 1,\ldots,2014$

$$(a'_i, b'_i) = (b_i, a_i) \text{ if } (a_i, b_i) \neq (1,2) \text{ and } (a'_i, b'_i) = (a_i, b_i) \text{ otherwise.}$$

(Applying this transform twice we get the initial triple.)

Applying Φ_1, we get the property (P1) for (b,a): the number of entries 1,2 in A' is b and the number of entries 1,2 in B' is a. Let us check that (P2) will also be satisfied. If no, then there is i with $a'_i = 2$ and $b'_i \in \{2,1\}$; the pair (2,2) cannot occur since we interchanged (a_i, b_i); b'_i cannot be 1 since we did not interchange (1,2). As to the sequence C, if $b'_i = 2$, then a_i was equal to 2 which gives that $c_i = 0$. So, $f(a,b,c) = f(b,a,c)$.

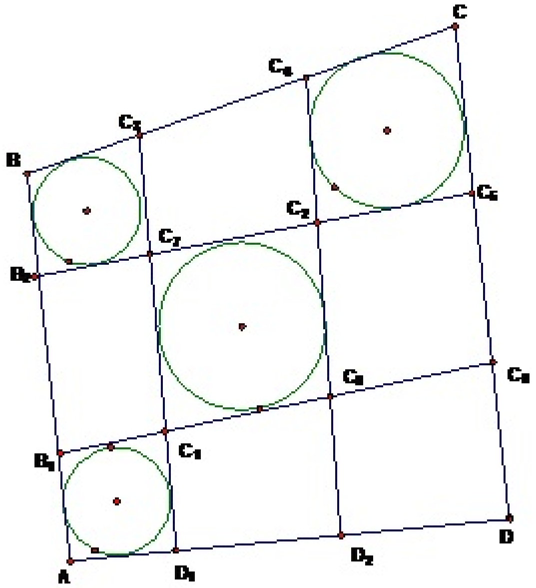
To prove now that $f(a,b,c) = f(a,c,b)$ we consider a similar bijection Φ_2:

$\Phi_2((A, B, C)) = (A, C', B')$ with $B' = (b'_1, \ldots, b'_{2014})$, $C' = (c'_1, \ldots, c'_{2014})$ and for all $i = 1,\ldots,2014$

$$(b'_i, c'_i) = (c_i, b_i) \text{ if } (b_i, c_i) \neq (1,2) \text{ and } (b'_i, c'_i) = (b_i, c_i) \text{ otherwise.}$$

Using a similar argument as explained above, conditions for (P1), (P2) hold for a pair (B', C'). To show a full check with (P2), finally note that if $a_i = 2$, then $b_i = c_i = 0$ and the same holds after Φ_2.

Problem 6.



We use the following

Lemma. A convex quadrilateral $XYZT$ has an inscribed circle if and only if
$$\tan\frac{\angle YXZ}{2} : \tan\frac{\angle TXZ}{2} = \tan\frac{\angle YZX}{2} : \tan\frac{\angle TZX}{2}.$$

Proof of the lemma. Let the incircles of triangles XYZ and XTZ touch XZ at Y_1 and T_1, respectively. It is easy to see that $XY_1 = \frac{XY + XZ - YZ}{2}$ and $XT_1 = \frac{XT + XZ - YZ}{2}$, and $XYZT$ is tangential if and only if $XY_1 = XT_1$, which is equivalent to
$XY_1 : Y_1Z = XT_1 : T_1Z$ and, further, to
$$\tan\frac{\angle YXZ}{2} : \tan\frac{\angle TXZ}{2} = \tan\frac{\angle YZX}{2} : \tan\frac{\angle TZX}{2}.$$

Applying the lemma to quadrilaterals $B_1AD_1C_1$, $C_7C_1C_8C_2$, $C_4C_2C_6C$, we have
$$\tan\frac{\angle B_1AC_1}{2} : \tan\frac{\angle D_1AC_1}{2} = \tan\frac{\angle B_1C_1A}{2} : \tan\frac{\angle D_1C_1A}{2} = \tan\frac{\angle C_8C_1C_2}{2} : \tan\frac{\angle C_7C_1C_2}{2} =$$
$$= \tan\frac{\angle C_8C_2C_1}{2} : \tan\frac{\angle C_7C_2C_1}{2} = \tan\frac{\angle CC_2C_4}{2} : \tan\frac{\angle CC_2C_5}{2} = \tan\frac{\angle C_2CC_4}{2} : \tan\frac{\angle C_2CC_5}{2}.$$
that is,
$$\tan\frac{\angle BAC}{2} : \tan\frac{\angle DAC}{2} = \tan\frac{\angle BCA}{2} : \tan\frac{\angle DCA}{2}$$
and the quadrilateral $ABCD$ is circumscribed.

Applying again the lemma to quadrilaterals $BB_2C_7C_3$, $C_7C_1C_8C_2$, $ABCD$, we get
$$\tan\frac{\angle C_6C_8D}{2} : \tan\frac{\angle D_2C_8D}{2} = \tan\frac{\angle C_6DC_8}{2} : \tan\frac{\angle D_2DC_8}{2}.$$
and the quadrilateral $C_6C_8DD_2$ is circumscribed, q.e.d.