Problem 1.

Let KMN and $KM'N'$ be two beautiful triangles with common vertex, $\angle KMN = \angle A = \angle KM'N'$. Without loss of generality, assume that M' lies between B and M. The segments MN and $M'N'$ have a common point, we denote it by R. Since $\angle KMR = \angle KM'R$, the points K, M, M', R are concyclic and $\angle KM'M = 180^0 - \angle KRM = \angle KRN$. Similarly, K, N, N', R are concyclic, therefore $\angle KN'N = \angle KRN$. Thus $\angle KM'C = \angle KM'M = \angle KN'N = 180^0 - \angle KN'C$. It follows that the quadrilateral $KM'CN'$ is cyclic, and $180^0 = \angle C + \angle M'KN' = 2\angle C$, so the angle C is right.

Problem 2.

Does there exist a function $f: R \to R$ satisfying the following two conditions:

1) f takes all real values;
2) $f(f(x)) = (x-1)f(x) + 2$ for all $x \in R$?

Answer: there is no such f .

Suppose that such f does exist.

1. Denote $f(1) = a$. Set $x = 1$ in
$$f(f(x)) = (x-1)f(x) + 2 \,, \qquad (1)$$

Then *f(a)*=2.

2. Now setting $x = a$ in (1) we obtain $f(2) = (a-1) \cdot 2 + 2$, then $f(2) = 2a$.
3. By condition, $\exists b \in R | f(b) = 1$. Let $x = b$ in (1), then
$$a = f(1) = f\big(f(b)\big) = (b-1) \cdot 1 + 2 = b + 1 \quad , b = a - 1.$$
4. Further, , $\exists c \in R | f(c) = 0$. Setting $x = c$ in (1) we obtain
$$f(0) = f\big(f(c)\big) = (c-1) \cdot 0 + 2 = 2, \; f(0) = 2.$$
So we have $2 = f(0) = f(a)$, whence $f\big(f(0)\big) = f(f(a))$, or
$(0-1)f(0) + 2 = (a-1)f(a) + 2$, or $-1 \cdot 2 + 2 = (a-1) \cdot 2 + 2$, hence $a = 0$.

As a result: $f(0) = 2, f(2) = 2a = 0, f(1) = 0, b = -1, f(-1) = 1$.

5. Let now $d \in R$ be such that $f(d) = -1$. Set $x = d$ in (1), then
$1 = f(-1) = f\big(f(d)\big) = (d-1) \cdot (-1) + 2 = -d + 3$ whence $d = 2$. That is $f(2) = -1$, contrary to $f(2) = 0$.

Note. There exist function f satisfying (1) such that $E(f) \neq R$. For example

$$f(x) = \begin{cases} \frac{x-2}{x-1}, & x \neq 1 \\ 0, & x = 1 \end{cases} \quad \text{or } f(x) = \begin{cases} 0, & x \neq 0 \\ 2, & x = 0 \end{cases}$$

Problem 3.

The answer is 180.

We reformulate the problem as follows. Given are 100 lattice points (that is, points with integral coordinates). How many pairs of neighbours (points at distance 1) can they form?

To prove that this problem is equivalent to the original one, we assign the number $2^i 3^j$ to the point (i, j). In the set of numbers thus obtained the number of pairs in question is equal to the number of neighbouring points in the set of 100 points.

Conversely, in any set of 100 numbers we find for each number its largest divisor m not divisible by 2 or 3 and divide the set into groups of numbers with the same m. Obviously the numbers in a good pair belong to the same group. Now we can assign to each group a set of points where a point (i, j) corresponds to the number $2^i 3^j m$. If some numbers from different groups correspond to coinciding or neighbouring points, we translate the image of each group by a vector long enough to avoid that.

We can prove now that the maximum number of neighbouring pairs is attained when the points form a 10×10 square (and then the number is 180).

Let us consider rows (i.e. the set of points with the same ordinate) and columns (i.e. the set of points with the same abscissa). Suppose we have a nonempty rows and b nonempty columns. Clearly $ab \geq 100$.

If a row contains k points then its points form at most $k-1$ pairs. Denoting the numbers of points in the rows by $k_1, k_2, ..., k_a$, we have at most $(k_1-1)+(k_2-1)+...+(k_a-1)=100-a$ horizontal pairs of neighbouring points. Similarly, we have at most $100-b$ pairs of vertical pairs. Adding these inequalities we have that the total number of pairs does not exceed $200-a-b \leq 200-2\sqrt{ab} \leq 180$.