6. Find all integer solutions of the equation $2x^2 - y^{14} = 1$.

The answer is $x = \pm 1$, $y = \pm 1$.

Solution.

Lemma 1. If $a > 1$ is an integer, then $a^6 - a^5 + a^4 - a^3 + a^2 - a + 1$ is not a perfect square.

Proof. If $a^6 - a^5 + a^4 - a^3 + a^2 - a + 1$ is a perfect square then so is $256(a+1)^2(a^6 - a^5 + a^4 - a^3 + a^2 - a + 1) = 256(a^8 + a^7 + a + 1)$. The latter however is impossible since

$$(16a^4 + 8a^3 - 2a^2 + a - 1)^2 < 256(a^8 + a^7 + a + 1) < (16a^4 + 8a^3 - 2a^2 + a)^2$$

for $a \geqslant 3$ and $a^6 - a^5 + a^4 - a^3 + a^2 - a + 1 = 43$ for $a = 2$.

Lemma 2. If a is an integer then $(a + 1, a^6 - a^5 + a^4 - a^3 + a^2 - a + 1)$ equals 1 or 7.

Proof. The difference

$$a^6 - a^5 + a^4 - a^3 + a^2 - a + 1 - 7 = (a^6 - 1) - (a^5 + 1) + (a^4 - 1) - (a^3 + 1) + (a^2 - 1) - (a + 1)$$

is divisible by $a + 1$, therefore, if $(a + 1, a^6 - a^5 + a^4 - a^3 + a^2 - a + 1) = d$ then 7 divides d.

Now in the original problem we have

$$2x^2 = (y^2 + 1)(y^{12} - y^{10} + y^8 - y^6 + y^4 - y^2 + 1).$$

Since 7 never divides $y^2 + 1$ for integral y, it follows from Lemma 2 that

$$(y^2 + 1, y^{12} - y^{10} + y^8 - y^6 + y^4 - y^2 + 1) = 1,$$

therefore one of the factors is a square and another is a square multiplied by 2.

However $y^{12} - y^{10} + y^8 - y^6 + y^4 - y^2 + 1$ is odd, thus

$$y^{12} - y^{10} + y^8 - y^6 + y^4 - y^2 + 1 = v^2.$$

We have $y^2 \leqslant 1$ by Lemma 1, $y = 0$ does not give an integral x, so $y = \pm 1$ and $x = \pm 1$.

Marking scheme

$y^{14} + 1$ is factored, factors proved to be coprime – 1 point

$y^{12} - y^{10} + y^8 - y^6 + y^4 - y^2 + 1$ is proved to be perfect square – 1 point

$y^{12} - y^{10} + y^8 - y^6 + y^4 - y^2 + 1$ lies between a^2 and $(a + 1)^2$ for some a – 1 point

4. Answer: there are not such m, n, f.

Proof. Let $f: R \to R$ be any function satisfying

$$f(f(x)) = 2f(x) - x - 2, \forall x \in R \quad (1).$$

We prove that there do not exist integers m, k such that $k \geq 0$ and

$$f(m) = m + k \qquad (2)$$

1. Let first $k = 0$, then (2) becomes $f(m) = m$.

Set $x = m$ in (1), then $m = f(m) = f(f(m)) = 2f(m) - m - 2 = 2m - m - 2$, or $m = m - 2$ which is impossible.

2. Let now $k = 1$, then we have

$$f(m) = m + 1 \ \to \ f(m+1) = f(f(m)) = 2f(m) - m - 2 = 2(m+1) - m - 2$$

or $f(m+1) = m$.

But then

$$m + 1 = f(m) = f(f(m+1)) = 2f(m+1) - (m+1) - 2 = 2m - m - 3 = m - 3,$$

a contradiction.

Suppose now that there is some integer $k \geq 2$ such that (2) is valid for some m. Choose the minimum possible such k.

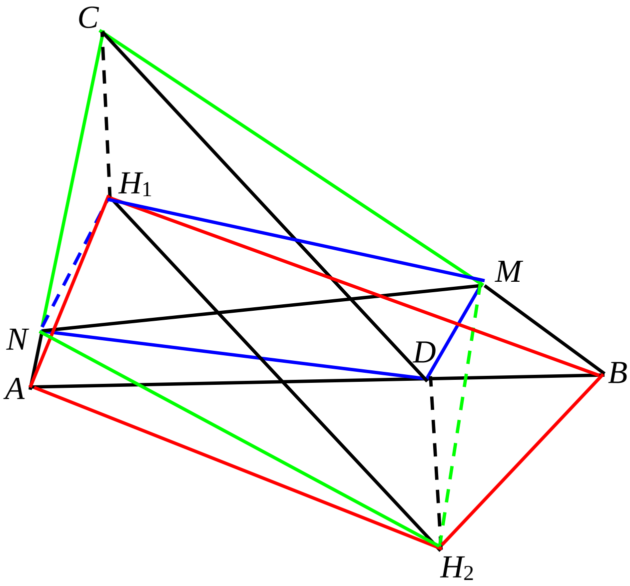
We have $f(m) = m + k \ \to$

$$f(m+k) = f(f(m)) = 2f(m) - m - 2 = 2(m+k) - m - 2 = (m+k) + (k-2).$$

Denote $m_1 = m + k$, $k_1 = k - 2$, then $f(m_1) = m_1 + k_1$. This contradicts with the minimality of k if $k_1 \geq 2$. But if $k_1 < 2$ then $k_1 = 0$ or $k_1 = 1$, which is impossible. Thus the proof is finished.

Problem 1. Given is an acute ΔABC. Let $D \in AB$, $DM \perp BC$ ($M \in BC$) and $DN \perp AC$ ($N \in AC$). If H_1 and H_2 are the orthocentres of ΔMNC and ΔMND respectively, prove that the area of the quadrilateral AH_1BH_2 does not depend on the position of D on the side AB.

Solution: Since $NH_1 \perp BC$ (altitude) and $DM \perp BC$ (given), then $NH_1 \parallel DM$. Also, $MH_1 \perp AC$ (altitude) and $DN \perp AC$ (given). Thus $MH_1 \parallel DN$ and it follows that $NDMH_1$ is a parallelogram. Analogously $NCMH_2$ is a parallelogram and consequently $CN = MH_2$.



On the other hand $CH_1 \perp MN$ and $DH_2 \perp MN$, which gives that $CH_1 \parallel DH_2$. Since the sides of ΔCNH_1 are parallel to the sides of ΔH_2MD and $CN = MH_2$ it follows that $\Delta CNH_1 \cong \Delta H_2MD$. Thus $CH_1 = DH_2$, which together with $CH_1 \parallel DH_2$ gives that CH_1H_2D is a parallelogram too. Consequently $H_1H_2 = CD$.

If $\angle ADC = \varphi$, then $S_{ABC} = \dfrac{1}{2} AB.CD \sin\varphi$ and finally $S_{AH_1BH_2} = \dfrac{1}{2} AB.H_1H_2 \sin\varphi = \dfrac{1}{2} AB.CD \sin\varphi = S_{ABC}$, which ends the proof.

<u>*Marking scheme*</u>. 5 points to prove that $H_1H_2 = CD$ and 2 points to end the proof.
Partial credits: 3 points to prove that $\Delta CNH_1 \cong \Delta H_2MD$ (1 point to prove that $NDMH_1$ is a parallelogram and 1 point to prove that $NCMH_2$ is a parallelogram).

Problem 1. Given is an acute ΔABC. Let $D \in AB$, $DM \perp BC$ ($M \in BC$) and $DN \perp AC$ ($N \in AC$). If H_1 and H_2 are the orthocentres of ΔMNC and ΔMND respectively, prove that the area of the quadrilateral AH_1BH_2 does not depend on the position of D on the side AB.

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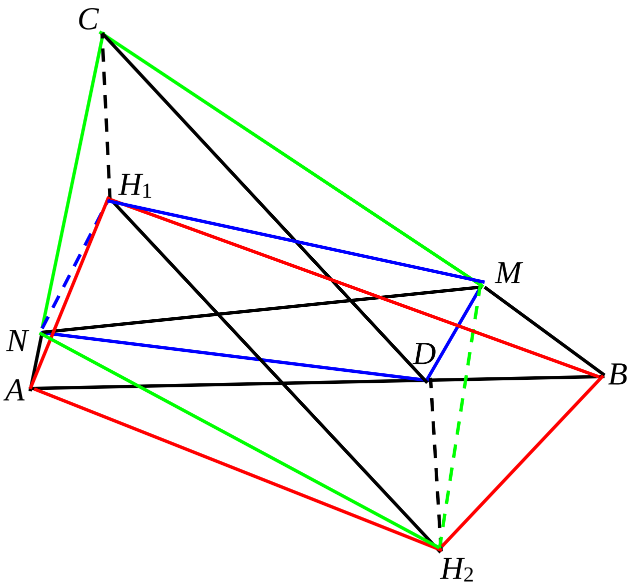

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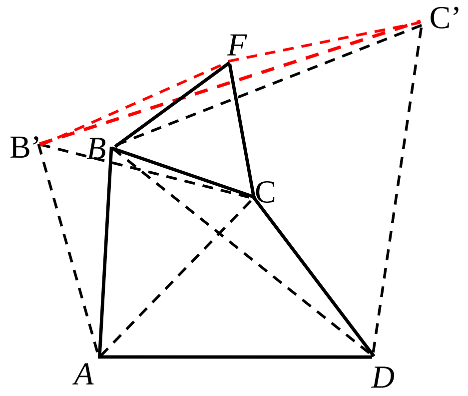
Задача 5. На диагоналях выпуклого четырехугольника *ABCD* построены правильные треугольники *ACB´* и *BDC´*, причем точки *B* и *B´* лежат по одну сторону от *AC*, а точки *C* и *C´* лежат по одну сторону от *BD*. Найдите $\angle BAD + \angle CDA$, если известно, что $B'C' = AB + CD$.

(*Армения*)

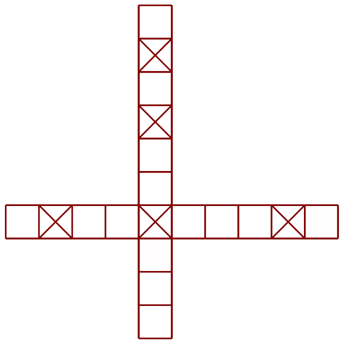
Первое решение: Построим равносторонний треугольник BCF, как показано на чертеже. Тогда $\angle FBC = 60^0 = \angle C'BD$ и следовательно $\angle FBC' = \angle CBD$. Так как $BF = BC$ и $BC' = BD$, имеем $\Delta BFC' \cong \Delta BCD$. Откуда $FC' = CD$ и $\angle BFC' = \angle BCD$. Аналогично, $B'F = AB$ и $\angle B'FC = \angle ABC$. Из равенства $B'C' = AB + CD$ следует равенство $B'C' = B'F + FC'$ значит точка F лежит на отрезке $B'C'$. Но тогда $\angle B'FC + \angle BFC' = 180^0 + \angle BFC = 240^0$.



Получаем, что $\angle BCD + \angle ABC = \angle BFC' + \angle B'FC = 240^0$ и, следовательно, $\angle BAD + \angle CDA = 120^0$.

Alternative proof of the estimation $m \leq 4n - 8$. (*)

First we call a cross a union of some row and some column of the table such that the row as well as the column contains ≥ 3 unit squares from the given convenient set M, and, besides, one of the squares is common for this row and this column (see fig. below)



Then the given statement will immediately follow the next Claim.

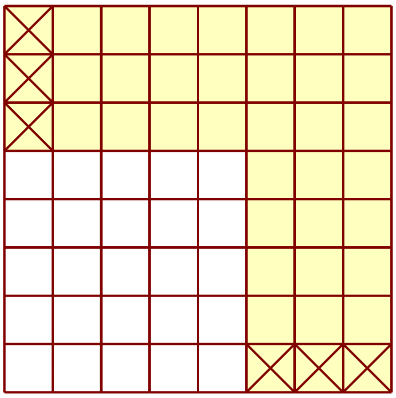
Claim. If some set M contains m $\geq 4n - 7$ squares of the table then there exists some cross in the table.

To prove this we use induction on n.

Base. $n = 5$, 6 or 7. Consider one of the rows or the columns of the table that contains the maximum k of squares from M among all the rows and columns. Note that $k \geq 3$. Let for definiteness these k squares are at the very top of the leftmost column. Any of the top k rows contains ≤ 2 squares from M, otherwise Claim would be proved. Hence, the total number of squares from M in the top k rows is at most $2k$. The remaining $n - k$ rows contains in total at most $k(n - k)$ squares. It follows that $4n-7 \leq m \leq 2k + k(n - k)$, or $k^2 - (n+2)k + 4n-7 \leq 0$. However, the discriminant of the LHS is $D = (n+2)^2 - 4(4n-7) = (n-4)(n-8) < 0$ for $n = 5$, 6 and 7, hence $k^2 - (n+2)k + 4n-7 > 0$, contradiction.

Suppose now that Claim is true for $n = k$ and prove that it is true also for $n = k + 3$.

First, there is a row containing at least 3 squares from the given set M (otherwise $m \leq 2n$). Similarly, there is a column containing ≥ 3 squares from M. Consider such a row and such a column. WLOG we can assume that this column is the leftmost one in the table and the squares from M in it are all at the very top of column; similar assumption for the row (see fig. below).



Consider one of the top three rows. If it contains at least 3 squares from M then this row together with the leftmost column give us the needed cross. Hence we may assume that any of the three top rows as well as any of the three rightmost columns contains at most 2 squares from M. Thus we have in total at most 12 squares from M in the shaded area.

Now, since $m \geq 4n - 7 = 4(k + 3) - 7 = 4k - 7 + 12$, we conclude that the remaining $k \times k$ table contains at least $4k - 7$ squares from M. Hence, by the induction hypothesis, this $k \times k$ table contains a cross which is also a cross for the initial $n \times n$ table. This finishes the proof of the Claim.