Marking Scheme

of the 1st day problems in Mathematics

Problem No 1. Answer: $p = 13, q = 3$.

1st solution. First, notice that $p^3 - 1 > p^3 - p > q(q^6 - 1) > q^6 - 1 > q^3 - 1$, hence $p > q$. It follows that $p^3 - q^7 = p - q > 0$ and $p > q^{\frac{7}{3}}$.

Further, from the original equation we obtain

$$q(q^2-1)(q^2-q+1)(q^2+q+1) = p(p^2-1). \qquad (*)$$

The primality of p together with $p > q$ implies either $q^2 - 1 \vdots p$ or $q^2 - q + 1 \vdots p$ or $q^2 + q + 1 \vdots p$. Anyway, we have $q^2 + q + 1 \geq p > q^{\frac{7}{3}}$.

Suppose that $q \geq 5$. Then $q > 1{,}5^3$, so that $q^{\frac{7}{3}} > 1{,}5q^2$ and $q^2 + q + 1 > 1{,}5q^2$, hence $0{,}5q^2 - q - 1 < 0$. So we obtain $q < 1 + \sqrt{3}$ which contradicts $q \geq 5$.

Thus we have $q \leq 3$. **(5 points)**

1. If $q = 2$ then $p^3 - p = 126$. It is easy to check that the cases $p = 2, 3, 5$ are not suitable. Now, for $p \geq 7$ we have $p^3 - p \geq 7^3 - 7 > 126$, because the function $x^3 - x$ is increasing for $x \geq \frac{1}{\sqrt{3}}$. Thus the case $q = 2$ is not possible.

2. If $q = 3$ then $p^3 - p = 2184$. One can easily verify that the value $p = 13$ is appropriate, and for $p > 13$ we have $p^3 - p > 2184$. **(+2 points)**

2nd solution. As in the 1st solution, we have $p > q^{\frac{7}{3}} > q^2$, so that $p > q, p > q^2 - q + 1$. Therefore $q^2 + q + 1 \vdots p$. Further, $2p > 2q^2 > q^2 + q + 1$ hence

$$q^2 + q + 1 = p \qquad (1).$$ **(5 points)**

Then (*) becomes $q(q-1)(q^2-q+1) = (q^2+q)(q^2+q+2)$, or $q^3 - 2q^2 + 2q - 1 = q^2 + q + 2$, i.e. $q^3 - 3q^2 + q - 3 = (q^2+1)(q-3) = 0$, $q = 3$. Then from (1) $p = 3^2 + 3 + 1 = 13$. **(+2 points)**

Problem No 2
Solution.

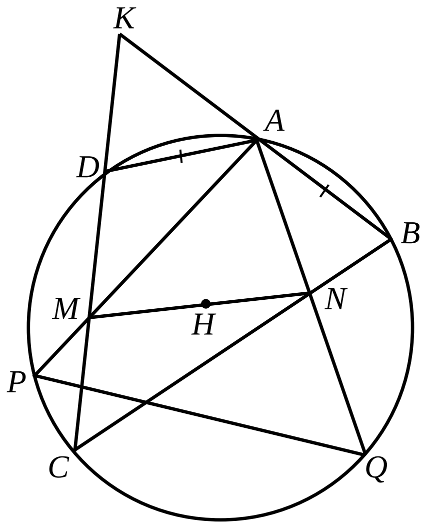

Fig. 1

Choose a point K on the extension of MD beyond D such that $DK = NB$ (see Fig. 1). It follows from $\angle KDA = 180^0 - \angle ADC = \angle ABN$, $DA = AB$, and $DK = NB$ that $\Delta KDA = \Delta NBA$. Therefore $KA = AN$ and $MK = MD + DK = MD + NB = MN$. Hence, $\Delta KMA = \Delta NMA$ and $\angle DMA = \angle NMA$. Similarly we can prove that $\angle MNA = \angle BNA$.

Take a point H on the segment MN such that $MH = MD$. Then $NH = BN$. Since $\angle DMA = \angle HMA$ and $MD = MH$, the points D and H are symmetric with respect to AP. Similarly, B and H are symmetric with respect to AN. Hence $\angle DAB = 2\angle MAN$. Therefore,

$$\angle HPA = \angle DPA = \angle ABD = 90^0 - \frac{1}{2}\angle DAB = 90^0 - \angle MAN .$$

It means that $PH \perp AQ$. In the same way we can prove that $QH \perp AP$. Thus the altitudes of APQ meet at H.

Marking scheme:
If correctly proved that triangles *ADM* and *AHM* are equal (or similar) – (4 points)
If correctly proved that *BH* and *AQ* are perpendicular lines (or similar) – (+ 3 points)
Statement that *BP* and *AQ* are perpendicular, without proof – (1 point)

Problem No. 3

First solution.

Note that every figure consists of two right isosceles triangles with leg 1. The hypotenuse of such triangle may be directed either from lower left to upper right corner of a square (we call it *left hypotenuse*) or from upper left to lower right corner (we call it *right hypotenuse*). Since each hypotenuse belongs to two triangles, the number of triangles with left hypotenuse is even. On the other hand, every figure of the first kind contains exactly one triangle with left hypotenuse, while every other figure contains an even number of such triangles. It follows that the number of figures of the first kind (big triangles) is even.

Now we colour the columns of the rectangle in black and white: columns with even numbers will be black and the rest will be white. Figures of the first and third kind can lie in a single column (in this case (we call them *vertical*) or intersect two neighbouring columns (then they are *horizontal*). Obviously the black part of a figure of the second kind or a vertical figure has integral area. The black part of a horizontal figure always has area 1/2. Since the whole black area is integral, the total number of horizontal figures is even.

Applying the same argument to rows we prove that the total number of vertical figures is also even. Thus we have proved that the total number of figures of the first and the third kind is even, and the number of figures of the first kind is also even, whence the desired result follows.

Second solution.

We begin with proving that the number of figures of the first kind is even in the same way as in the above solution. Then we colour our rectangle black and white as a chessboard, that is, so that unit squares sharing a side are of different colours. Every figure of the second kind covers an integral black area. The black part of every other figure has area $1\over 2$. Thus the total number of triangles and parallelograms is even, and it remains to subtract the even number of triangles.

Third solution.

The figures of the third kind (parallelograms) can be distributed into 2 groups: *horizontal* (covered by one row) and *vertical* (covered by one column). First we consider a *horizontal colouring* of the rectangle, as shown in the Fig. 2 below.

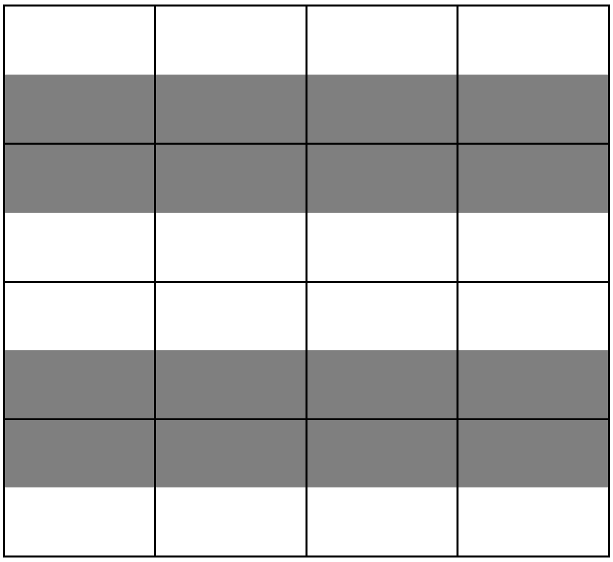

Fig. 2

Obviously the black and white areas of this colouring are equal.

Note that in every figure of the second kind (we shall call it merely a square) both black and white parts always have area 1/2. On the other hand, in a figure of the first kind (a triangle) as well as in a vertical parallelogram black and white areas are either 3/4 and 1/4 respectively (then we say that the figure is *almost black*) or 1/4 and 3/4 respectively (and we call it *almost white*). Suppose we have a squares, b horizontal parallelograms, c vertical parallelograms and d triangles. We need to prove that $b + c$ is even. Let $c = c_1 + c_2$, $d = d_1 + d_2$, where c_1, c_2, d_1, d_2 are numbers of almost white vertical parallelograms, almost black vertical parallelograms, almost white triangles, and almost black triangles, respectively.

Since black and white parts of the rectangle have equal areas,

$$\tfrac12 a + \tfrac12 b + \tfrac34 c_1 + \tfrac14 c_2 + \tfrac34 d_1 + \tfrac14 d_2 = \tfrac12 a + \tfrac12 b + \tfrac14 c_1 + \tfrac34 c_2 + \tfrac14 d_1 + \tfrac34 d_2,$$

therefore $c_1 + d_1 = c_2 + d_2$.

Using vertical colouring in the same way, we get $b_1 + d_3 = b_2 + d_4$, where b_1 and b_2 are the numbers of almost white and almost black horizontal parallelograms in the new colouring ($b_2 + b_2 = b$), d_3 and d_4 are the numbers of almost white and almost black triangles in the new colouring ($d_3 + d_4 = d = d_1 + d_2$).

Now we have $b + c = b_1 + b_2 + c_1 + c_2 = 2 b_2 + 2 c_2 + d_4 + d_2 - (d_1 + d_3) = 2 b_2 + 2 c_2 + 2d - 2 d_3 - 2 d_1$ is even, QED

Marking scheme:

the number of figures of the first kind is even – 3 points;

the total number of figures of the first and the third kind is even – 3 points.

Marking Scheme
of the 2nd day problems in Mathematics

Problem No 4.

Solution: The biggest prime, which could be reached when n is fixed, would not exceed the sum of the biggest odd positive integer smaller than n, all even integers from 2 to n and some additional 2-es. It follows that if $n=14$, then the biggest prime number, which could be achieved, will not exceed the number $13+2+4+6+8+10+12+14+3\cdot 2=75$. On the other hand the operation under consideration is invariant with respect to the parity of the number of the odd positive integers which do not exceed n. It follows that if $n=15$ or $n=16$, the integer 97 could not be the last. Let $n=17$. The sum of all even positive integers less than 17 is equal to 72. The odd positive integers less than 17 give four additional 2-es when the operation is applied to them in pairs. Since $97-(72+2\cdot 4)=17$, the only way to achieve 97, when $n=17$, is to start by 17 and in a suitable order to add integers from the set $\{2, 2, 2, 2, 2, 4, 6, \ldots, 16\}$, obtaining each time a new prime number different from the previous one. Two of these 12 integers in the set are equal to 0 modulo 3, three of them are equal to 1 modulo 3 and seven of them are equal to 2 modulo 3. The number 17 is equal to 2 modulo 3 and a number equal to 2 or to 0 modulo 3 should be added to it only. When a number equal to 1 modulo 3 is obtained, then a number equal to 1 or to 0 modulo 3 should be added only. Thus 97 could not be achieved since the integers in the set under consideration which are equal to 1 modulo 3 are less than the integers which are equal to 2 modulo 3. The answer of the problem is $n=18$. Firstly, apply the operation to the pairs (3,5); (7,9); (11,13) and (15,17). Further; proceed in the following way: (1,2) $\rightarrow$ 3; (3,2) $\rightarrow$ 5; (5,2) $\rightarrow$ 7; (7,4) $\rightarrow$ 11; (11,2) $\rightarrow$ 13; (13,6) $\rightarrow$ 19; (19,10) $\rightarrow$ 29; (29,8) $\rightarrow$ 37; (37,16) $\rightarrow$ 53; (53,14) $\rightarrow$ 67; (67,12) $\rightarrow$ 79, (79,18) $\rightarrow$ 97.

Marking scheme:

1 points for $n \geq 14$

2 points for a correct proof $n \geq 17$, 1 point for the invariance of the operation (this point is not accumulated);

2 points for a correct consideration of the case $n=17$;

2 points for a correct example in the case $n=18$.

Problem No 5. In every vertex of a regular *n*-gon exactly one chip is placed. At each step one can exchange any two neighbouring chips. Find the least number of steps necessary to reach the arrangement where every chip is moved by $\left[\frac{n}{2}\right]$ positions clockwise from its initial position.

Solution.

The answer is $\left[\frac{n}{2}\right]\cdot\left[\frac{n+1}{2}\right]$.

To prove it we define some terms. We suppose that the chips are numbered 1, 2, ..., *n* and initially arranged so that their numbers increase clockwise. A *place* is a point where a chip stands; its *number* is the number of the chip standing there in the initial arrangement. We can describe an arrangement by a sequence of numbers of the chips standing on the places 1, 2, ..., *n*.

We say that k-th chip is *moved clockwise* if it is exchanged with its neighbour in clockwise direction, and *counterclockwise* if it is exchanged with its neighbour in counterclockwise direction.

I. The example of $\left[\frac{n}{2}\right]\cdot\left[\frac{n+1}{2}\right]$ steps giving the desired arrangement is constructed as follows: for every $k = \left[\frac{n+1}{2}\right], \left[\frac{n+1}{2}\right]-1, \ldots, 2, 1$ the *k*-th chip is moved clockwise $\left[\frac{n}{2}\right]$ times. When *k*-th chip is moved $\left[\frac{n}{2}\right]$ times, the arrangement is 1, 2, $k-1$, $\left[\frac{n+1}{2}\right]+1$, $\left[\frac{n+1}{2}\right]+2$, …, *n*, *k*, *k*+1, …, $\left[\frac{n+1}{2}\right]$. When we finally move the first chip, the desired arrangement is reached.

II. To prove this number is minimum, we define *total shift* of every chip as follows: it is 0 in the initial position and increases or decreases by 1 when a chip moves clockwise or counterclockwise, respectively.

When two chips are exchanged, their total shifts are changed by 1 and -1, therefore the sum of total shifts of all the chips is 0. Note that absolute value of total shift of every chip in the final arrangement is at least $\left[\frac{n}{2}\right]$. Obviously there are at least $\left[\frac{n+1}{2}\right]$ chips with total shift of the same sign. These chips together moved at least $\left[\frac{n}{2}\right]\cdot\left[\frac{n+1}{2}\right]$ in the same direction. But only one chip is moved in this direction at each step, therefore the number of steps is at least $\left[\frac{n}{2}\right]\cdot\left[\frac{n+1}{2}\right]$, QED.

Marking scheme:

An example for even n – 1 point;
An example for odd n – 1 point;
The inequality for even n – 2 points;
The inequality for odd n – 3 points.

Problem No 6.

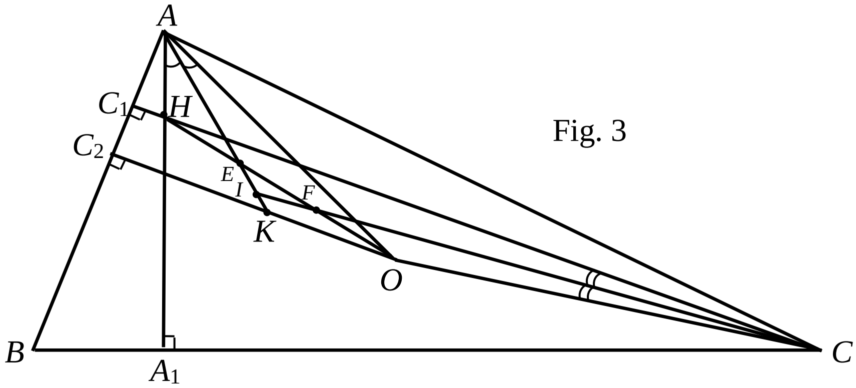

Fig. 3

Будем считать, что $\angle A > \angle B > \angle C$. Докажем, что $\angle OIH > 180^0 - \frac{1}{2}\angle A$. Тогда, так как $\angle A < 90^0$, то $180^0 - \frac{1}{2}\angle A > 135^0$ и следовательно $\angle OIH > 135^0$. В треугольнике ABC проведем высоты AA_1 и CC_1. Так как $\angle OAB = 90^0 - \angle C > 90^0 - \angle B = \angle BAA_1$ и $\angle OCA = 90^0 - \angle B > 90^0 - \angle A = \angle ACC_1$, точка O находится внутри треугольника A_1HC. Заметим, что $\angle BAA_1 = \angle OAC = 90^0 - \angle B$ и $\angle ACC_1 = \angle OCB = 90^0 - \angle A$. Следовательно, $\angle HAI = \angle IAO$ и $\angle HCI = \angle ICO$. Пусть прямые AI и CI пересекают отрезок HO в точках E и F соответствено, C_2 – середина стороны AB и K – точка пересечения прямых OC_2 и AI. Так как $\frac{HE}{EO} = \frac{AH}{AO} < \frac{HC}{AO} = \frac{HC}{CO} = \frac{HF}{FO}$ и прямые CI и OC_2 пересекаются на описанной окружности треугольника ABC, точка I находится на отрезке EK.

Если $\angle B \geq 60^0$, то $\angle AIC = 90^0 + \frac{1}{2}\angle B \geq 180^0 - \angle B = \angle AHC$. Заметим, что точка I находится внутри окружности, проходящей через точки A, H и C. Тогда $\angle AIH > \angle ACH = 90^0 - \angle A$ и

$$\angle OIH = \angle OIA + \angle AIH > \angle OKA + \angle AIH = 90^0 + \frac{1}{2}\angle A + \angle AIH > 180^0 - \frac{1}{2}\angle A.$$

Значит $\angle OIH > 180^0 - \frac{1}{2}\angle A$.

Если $\angle B < 60^0$, то $\angle AIC = 90^0 + \frac{1}{2}\angle B > 2\angle B = \angle AOC$. Следовательно, точка I находится внутри окружности, проходящей через точки A, O и C. Заметим, что $\angle IOA < \frac{1}{2}\angle C$. Тогда $\angle OIA = 180^0 - \angle IAO - \angle IOA =$

$$= 180^0 - \frac{1}{2}(\angle B - \angle C) - \angle IOA > 180^0 - \frac{1}{2}(\angle B - \angle C) - \frac{1}{2}\angle C = 180^0 - \frac{1}{2}\angle B.$$

Значит $\angle OIA > 180^0 - \frac{1}{2}\angle B$ и $\angle OIH > \angle OIA > 180^0 - \frac{1}{2}\angle B > 180^0 - \frac{1}{2}\angle A$.

Следовательно, $\angle OIH > 180^0 - \frac{1}{2}\angle A$.

Замечание. Эту оценку улучшить нельзя, так как при $\angle A = 90^0$, $\angle B \to 90^0$ имеем $\angle IAO = \frac{1}{2}(\angle B - \angle C) \to 45^0$, $\angle IOA \to 0^0$ и тогда $\angle OIH = \angle OIA \to 135^0$.

English version:

(1). Assume without loss of generality that $\angle A > \angle B > \angle C$. We will prove that $\angle OIH > 180^0 - \frac{1}{2}\angle A$ and it follows from $\angle A < 90^0$ and $180^0 - \frac{1}{2}\angle A > 135^0$, that $\angle OIH > 135^0$. Let AA_1 ($AA_1 \in BC$) and CC_1 ($CC_1 \in AB$) be altitudes of ΔABC. The conditions $\angle OAB = 90^0 - \angle C > 90^0 - \angle B = \angle BAA_1$ and $\angle OCA = 90^0 - \angle B > 90^0 - \angle A = \angle ACC_1$ imply that O is an interior point of ΔA_1HC. Note, that $\angle BAA_1 = \angle OAC = 90^0 - \angle B$ and $\angle ACC_1 = \angle OCB = 90^0 - \angle A$. Then $\angle HAI = \angle IAO$ and $\angle HCI = \angle ICO$. Let E and F be the common points of the lines AI and CI and the segment HOq respectively. Let also C_2 be the midpoint of the side AB and K be the common point of the lines OC_2 and AI. Since $\frac{HE}{EO} = \frac{AH}{AO} < \frac{HC}{AO} = \frac{HC}{CO} = \frac{HF}{FO}$ and the common point of the lines CI and OC_2 lies on the circumcircle of ΔABC, the point I is in the segment EK. We consider 2 cases.

(2). If $\angle B \ge 60^0$, then $\angle AIC = 90^0 + \frac{1}{2}\angle B \ge 180^0 - \angle B = \angle AHC$. Note, that I is an interior point of the circle through A, H and C. Then $\angle AIH > \angle ACH = 90^0 - \angle A$ and

$$\angle OIH = \angle OIA + \angle AIH > \angle OKA + \angle AIH = 90^0 + \frac{1}{2}\angle A + \angle AIH > 180^0 - \frac{1}{2}\angle A.$$

Thus, $\angle OIH > 180^0 - \frac{1}{2}\angle A$.

(3). If $\angle B < 60^0$, then $\angle AIC = 90^0 + \frac{1}{2}\angle B > 2\angle B = \angle AOC$. It follows that I is an interior point of the circle through A, O and C. Note, that $\angle IOA < \frac{1}{2}\angle C$. Then $\angle OIA = 180^0 - \angle IAO - \angle IOA =$

$$= 180^0 - \frac{1}{2}(\angle B - \angle C) - \angle IOA > 180^0 - \frac{1}{2}(\angle B - \angle C) - \frac{1}{2}\angle C = 180^0 - \frac{1}{2}\angle B.$$

It follows that $\angle OIA > 180^0 - \frac{1}{2}\angle B$ and $\angle OIH > \angle OIA > 180^0 - \frac{1}{2}\angle B > 180^0 - \frac{1}{2}\angle A$. Thus, $\angle OIH > 180^0 - \frac{1}{2}\angle A$.

Remark. The estimation could not be improved because if $\angle A = 90^0$ and $\angle B \to 90^0$, then $\angle IAO = \frac{1}{2}(\angle B - \angle C) \to 45^0$, $\angle IOA \to 0^0$ and it follows that $\angle OIH = \angle OIA \to 135^0$.

Marking scheme:

(1) – (2 points);
(2) or **(3)** – 3 points;
(2) + **(3)** – 5 points if the proofs are correct.