Problem 4. A quadratic trinomial $p(x)$ with real coefficients is given. Prove that there is a positive integer n such that the equation $p(x)=\frac{1}{n}$ has no rational roots.

Solution. Assume that a quadratic trinomial $p(x)=ax^2+bx+c$ gets all the values of the form $\frac{1}{n}$ at rational points. It is easy to see that the coefficients a,b,c are rational. Indeed, if $p(r_i)=\frac{1}{n_i}$ for i=1, 2, 3, where r_i are rational, then the numbers $a(r_1+r_2)+b=\frac{p(r_2)-p(r_1)}{r_2-r_1}$ and $a(r_2+r_3)+b=\frac{p(r_3)-p(r_2)}{r_3-r_2}$ are also rational, and so is their difference $a(r_1-r_3)$. It follows immediately that a, and therefore b and c are rational.

If a,b,c are represented as irreducible fractions there is a prime q that does not divide their denominators and numerators. When a rational r is written as an irreducible fraction $\frac{m}{n}$ and q^k, k>0, is the maximum power of q dividing n, the denominator of irreducible fraction equal to $p(r)=ar^2+br+c=\frac{am^2+bmn+cn^2}{n^2}$ is divisible by q^{2k} and not by q^{2k+1}. Therefore $p(r)$ cannot be equal to $\frac{1}{q^s}$ for any odd s. Since q does not divide the denominators of a,b,c, for n not divisible by q the equation $p\left(\frac{m}{n}\right)=\frac{1}{q^s}$ is also impossible.

Thus $n=q^s$ satisfies the condition for any odd s.

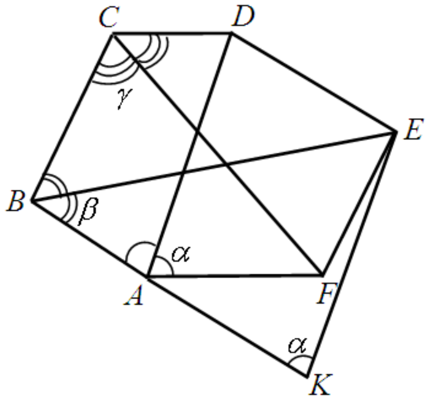
Problem 5. Given convex hexagon $ABCDEF$ with $AB \parallel DE$, $BC \parallel EF$, $CD \parallel FA$. The distance between the lines AB and DE is equal to the distance between the lines BC and EF and to the distance between the lines CD and FA. Prove that the sum $AD+BE+CF$ does not exceed the perimeter of hexagon $ABCDEF$.

Solution. First we prove the following statement

Lemma. Under the conditions of problem, $AD^2=(AB+DE)(FA+CD)$.

Indeed the point D is equidistant to the lines AB and FA. Hence $\angle BAD=\angle FAD$. Since $AB \parallel DE$, $CD \parallel FA$ we see that $\angle BAD=\angle FAD=\angle ADE=\angle ADC$. Similarly, $\angle ABE=\angle CBE=\angle FEB=\angle DEB$ and $\angle BCF=\angle DCF=\angle CFA=\angle CFE$.

Let $\angle BAD=\alpha$, $\angle ABE=\beta$ and $\angle BCF=\gamma$.



Since the sum of all internal angles of the hexagon equal to 720° we have $\alpha+\beta+\gamma=180^\circ$.

Consider a parallelogram $ADEK$. Clearly $BK = BA + AK = BA + DE$, $KE = AD$, $\angle AKE = \angle BAD = \alpha$, $\angle KBE = \angle ABE = \beta$, $\angle BEK = \gamma$.

Therefore, the triangles with sides $AD, AB+DE, BE$ and $FA+CD, AD, CF$ are similar. This implies that $AD^2 = (AB+DE)(FA+CD)$.

From Lemma, we get the following inequality:

$$AD = \sqrt{(AB+DE)(FA+CD)} \le \frac{AB+DE+FA+CD}{2}$$

Similarly,

$$BE \le \frac{AB+BC+EF+ED}{2}, \quad CF \le \frac{BC+CD+EF+FA}{2}.$$

From these three inequalities we obtain

$$AD + BE + CF \le AB + BC + CD + DF + EF + FA$$

Problem 6. A 10×10 table consists of 100 unit cells. A *block* is a 2×2 square consisting of 4 unit cells of the table. A set C of n blocks covers the table (i.e. each cell of the table is covered by some block of C) but no $n-1$ blocks of C cover the table. Find the largest possible value of n.

Solution. Consider an infinite table divided into unit cells. Any 2×2 square consisting of 4 unit cells of the table we also call a *block*.

Fix arbitrary finite set M of blocks lying on the table. Now we will consider arbitrary finite sets of unit cells of the table covered by M. For any such set Φ denote by $|\Phi|$ the least possible number of blocks of M that cover all cells from Φ.

We have the following properties.

1°. If $\Phi_1 \subseteq \Phi_2$ then $|\Phi_1| \le |\Phi_2|$.

2°. $|\Phi_1 \cup \Phi_2| \le |\Phi_1| + |\Phi_2|$.

3°. For the set A shown in the Fig.1, we have $|A| = 2$; for the set B shown in the Fig.2, we have $|B| = 3$.

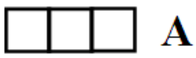


Fig. 1

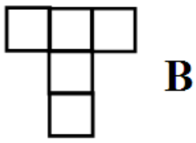


Fig. 2

4°. Let C be any rectangle 3×6 of the table. Then $|C| \le 10$.

This estimate is proved by consideration of different ways in which the cells X and Y (see Fig.3-8) can be covered by the blocks of M.

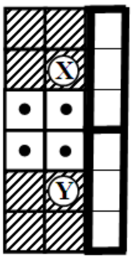


Fig. 3

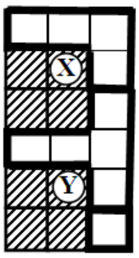


Fig. 4

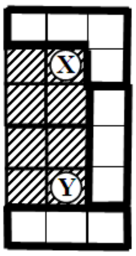


Fig. 5

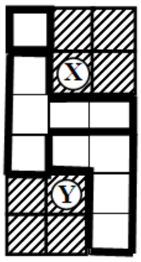


Fig. 6

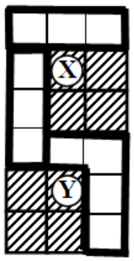


Fig. 7

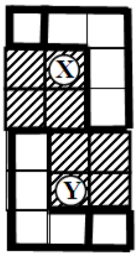


Fig. 8

For this figures we have , respectively, the following estimates:

Fig. 3 : Case 1) $|C| \le 2+2+3+1+1$ or Case 2) $|C| \le 1+1+1+1+1+1+1+1$;

Fig. 4 : $|C| \le 3+3+1+1+1$;

Fig. 5 : $|C| \le$ 2+2+3+1+1;

Fig. 6 : $|C| \le$ 3+3+1+1+1;

Fig. 7 : $|C| \le$ 2+2+3+1+1;

Fig. 8 : $|C| \le$ 3+3+1+1+1+1.

Remark 1. In the Fig 3. the first case means that the four marked cells are covered by at most 3 blocks; the second case means that the marked cells are covered by 4 different blocks.

Remark 2. The Fig 8. presents the only case where $|C|$ can attain the value 10; in all other figures we have in fact $|C| \le 9$.

5°. Let D be any 6×6 square of the table. From previous properties it follows that $|D| \le 20$. We claim that in fact $|D| \le 19$. This easily follows from the Fig.9 and remark 2 (using two different ways of dividing D into 2 rectangles 3×6).

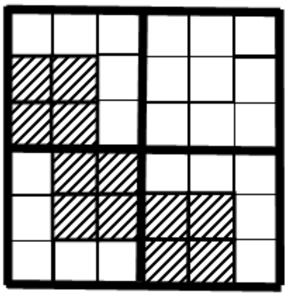


Fig. 9

Now we can finish the solution of the problem. Let E be given 10×10 table, D be its central 6×6 square. We have $|D| \le 19$. One can easily verify that $|E \setminus D| \le 20$ (applying the properties 1°-4°). So, $|E| \le |D| + |E \setminus D| \le 19 + 20 = 39$. On the other hand, Fig.10 shows that $n = 39$ can be attained.

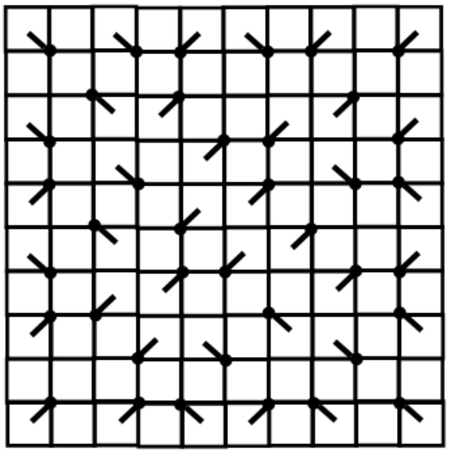


Fig. 10

In this Fig. the marked points are the centers of blocks participating in the covering.
For convenience we marked by half-diagonals those unit cells wich are covered by the only block.