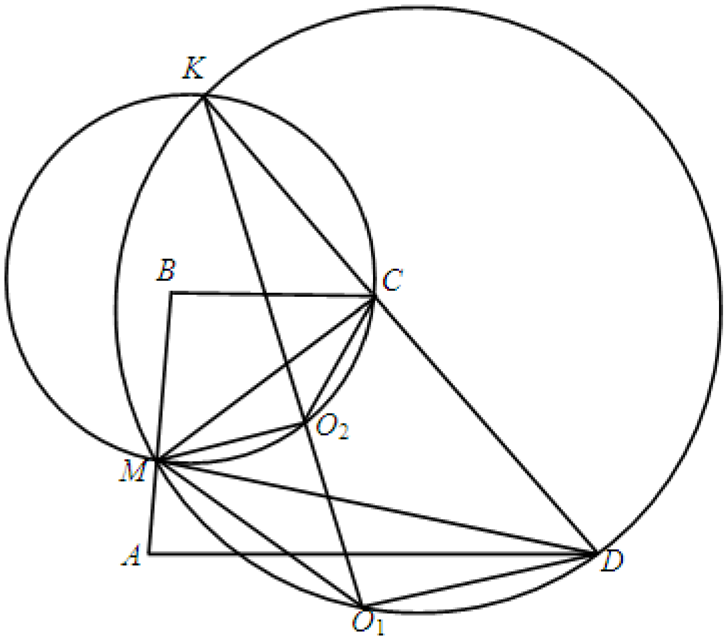
Problem 1. Given a trapezoid $ABCD$ ($AD \parallel BC$) with $\angle ABC > 90^\circ$. Point M is chosen on the lateral side AB. Let O_1 and O_2 be the circumcenters of the triangles MAD and MBC respectively. The circumcircles of the triangles MO_1D and MO_2C meet again at the point N. Prove that the line O_1O_2 passes through the point N.

Solution.



We have $\angle MO_2C = 360^\circ - 2\angle MBC = 2\angle MAD = \angle MO_1D$, hence $\Delta MO_1D \sim \Delta MO_2C$.

It follows that

$$\frac{MC}{MD} = \frac{MO_2}{MO_1} \qquad (1)$$

and $\angle CMO_2 = \angle O_1MD$.

Therefore

$$\angle O_1MO_2 = \angle DMC. \qquad (2)$$

From (1) and (2) we get

$$\Delta O_1MO_2 \sim \Delta DMC. \qquad (3)$$

Let the lines O_1O_2, CD intersect at point K.

From (3) it follows that $\angle MO_1K = \angle MDK$. Hence the points M, O_1, D, K lie on the same circle.

Thus $\angle CMO_2 = \angle O_1MD = \angle O_1KD$.

So, $\angle CMO_2 = \angle O_2KC$, i.e. the points M, O_2, C, K lie on the same circle.

It means that the points K and N coincide, i.e. the line O_1O_2 passes through the point N.

Problem 2. Find all odd positive integers $n > 1$ such that there is a permutation $a_1, a_2, ..., a_n$ of the numbers $1, 2, ..., n,$ where n divides one of the numbers $a_k^2 - a_{k+1} - 1$ and $a_k^2 - a_{k+1} + 1$ for each $k, 1 \le k \le n$ (we assume $a_{n+1} = a_1$).

Solution 1. Since $\{a_1, a_2, ..., a_n\} = \{1, 2, ..., n\}$ we conclude that $a_i - a_j \vdots n$ only if $i = j$.

From the problem conditions it follows that

$$a_{k+1} = a_k^2 + \varepsilon_k - nb_k, \qquad (1)$$

where $b_k \in Z$ and $\varepsilon_k = \pm 1$. We have $a_{k+1} - a_{l+1} = (a_k - a_l)(a_k + a_l) + (\varepsilon_k - \varepsilon_l) - n(b_k - b_l)$.

It follows that if $a_k + a_l = n$ then $\varepsilon_k \neq \varepsilon_l$ otherwise $a_{k+1} - a_{l+1} \vdots n$ - contradiction.
The condition $\varepsilon_k \neq \varepsilon_l$ means that $\varepsilon_k = -\varepsilon_l$.

Further, one of the a_i equals n. Let, say, $a_m = n$. Then the set $\{a_1, a_2, ..., a_n\} \setminus \{a_m\}$ can be divided into $\frac{n-1}{2}$ pairs (a_k, a_l) such that $a_k + a_l = n$. For any such pairs of indicies k, l we have $\varepsilon_k + \varepsilon_l = 0$.

Now add all the equalities (1) for $k = 1, 2, ..., n$. Then $\sum_{k=2}^{n+1} a_k = \sum_{k=1}^{n} a_k^2 - n\sum_{k=1}^{n} b_k + \varepsilon_m$, or $1 + 2 + ... + n = 1^2 + 2^2 + ... + n^2 - n\sum_{k=1}^{n} b_k + \varepsilon_m$ whence

$$n\sum_{k=1}^{n} b_k = \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} + \varepsilon_m = \frac{n(n+1)(n-1)}{3} + \varepsilon_m . \quad (2)$$

Note that if n is not divisible by 3 then the number $\frac{n(n+1)(n-1)}{3}$ is divisible by n (since $\frac{(n+1)(n-1)}{3}$ is integer). It follows from (2) that $\varepsilon_m \vdots n$ which is impossible.

Hence n is divisible by 3 and from (2) it follows that ε_m is divisible by the number $\frac{n}{3}$. The latter is possible only for $n = 3$ because $\varepsilon_m = \pm 1$. It remains to verify that $n = 3$ satisfies the problem conditions. Indeed, let $a_1 = 1, a_2 = 2, a_3 = 3$. Then $a_1^2 - a_2 + 1 = 0 \vdots 3$, $a_2^2 - a_3 - 1 = 0 \vdots 3$ and $a_3^2 - a_1 + 1 = 9 \vdots 3$.

Solution 2. Suppose that $a_1, a_2, ..., a_n$ is the desired sequence of residues modulo *n*, and let $f(a_i) = a_{i+1}$ (for $i = n$ we take $f(a_n) = a_1$).

The mapping f thus defined is a bijection. We have either $f(x) \equiv x^2 + 1 (\text{mod } n)$ or $f(x) \equiv x^2 - 1 (\text{mod } n)$. If $x^2 \equiv y^2 \equiv z^2 (\text{mod } n)$ then there are at most two different residues among $f(x), f(y), f(z)$, which is impossible. On the other hand, $f(x) \equiv f(-x) (\text{mod } n)$, therefore, if $x^2 \equiv y^2 (\text{mod } n)$ then either $x \equiv \pm y (\text{mod } n)$ or there is 0 among *x* and *y*. It follows that *n* cannot have two different prime factors (if it has, we can find *x* and *y* not divisible by *n* such that *n* divides (*x*+*y*)(*x*-*y*)). Furthermore, *n* cannot be divisible by a square of prime: if p^2 divides *n* then $0^2 \equiv \left(\frac{n}{p}\right)^2 \equiv \left(\frac{2n}{p}\right)^2 \equiv 0 (\text{mod } n)$. Thus *n* is prime.

We arrange all residues modulo *n*=2*k*+1 on a circle in the natural order 0, 1, …, *n*-1; there are *k*+1 squares and *k* non-squares among them.

Every residue must have a neighbouring square, therefore, there are no non-squares with difference 2. If $a^2 - b^2 \equiv 2 (\text{mod } n)$, i.e. $a^2 - 1 \equiv b^2 + 1 (\text{mod } n)$, then *f(a), f(-a), f(b), f(-b)* have only three possible values, and either *a* or *b* is 0. Thus every group of consecutive squares contains two residues (with only possible exception when it contains three, and 0 stands on the outside), and every group of consecutive non-squares contains two residues (with only possible exception when it contains only one next to 0). Since the number of squares exceeds that of non-squares by 1, exactly one of the two exceptions takes place.

In the first case 0, 1, 2 are squares, and 3, 4 are non-squares, a contradiction.

In the second case all the residues of the form 4*k*<*n* and 4*k*+1<*n* are squares and the rest are non-squares. For *n*>6 this means that 2, 3, 6 are non-squares which is impossible (since a non-square multiplied by non-zero square is a non-square, the product of two non-squares must be a square).

Thus $n \le 5$. Immediate calculation shows that n=3.

Problem 3. Let $a,b,c,d > 0$ and $abcd = 1$. Prove that

$$\frac{(a-1)(c+1)}{1+bc+c}+\frac{(b-1)(d+1)}{1+cd+d}+\frac{(c-1)(a+1)}{1+ad+a}+\frac{(d-1)(b+1)}{1+ab+b}\ge 0 .$$

Solution 1. It's easy to see that the needed inequality is equivalent to following one

$$\frac{ac+a+bc}{1+bc+c}+\frac{bd+b+cd}{1+cd+d}+\frac{ac+c+ad}{1+ad+a}+\frac{db+d+ab}{1+ab+b}\ge 4 . \qquad (1)$$

Since

$$ac+a+bc=\frac{(c+1)^2}{\frac{c+1}{a}}+\frac{(bc)^2}{bc}\ge\frac{(c+1+bc)^2}{\frac{c+1}{a}+bc}=\frac{ad(c+1+bc)^2}{cd+d+1},$$

or $\dfrac{ac+a+bc}{1+bc+c}\ge\dfrac{ad(c+1+bc)}{cd+d+1}$.

Hence

$$\frac{ac+a+bc}{1+bc+c}+\frac{bd+b+cd}{1+cd+d}+\frac{ac+c+ad}{1+ad+a}+\frac{db+d+ab}{1+ab+b}\ge$$

$$\ge\frac{ad(c+1+bc)}{cd+d+1}+\frac{ba(d+1+cd)}{da+a+1}+\frac{cb(a+1+da)}{ab+b+1}+\frac{dc(b+1+ab)}{bc+c+1}\ge$$

$$\ge 4\sqrt[4]{\frac{ad(c+1+bc)}{cd+d+1}\cdot\frac{ba(d+1+cd)}{da+a+1}\cdot\frac{cb(a+1+da)}{ab+b+1}\cdot\frac{dc(b+1+ab)}{bc+c+1}}=4.$$

QED.

Solution 2.

Since $abcd = 1$, there exist positive real numbers x, y, z, t, such that

$$a=\frac{x}{y},\ b=\frac{y}{z},\ c=\frac{z}{t},\ d=\frac{t}{x} .$$

Then inequality can be rewritten in the form

$$\frac{(x-y)(z+t)}{y(y+z+t)}+\frac{(y-z)(x+t)}{z(z+x+t)}+\frac{(z-t)(x+y)}{t(z+x+y)}+\frac{(t-x)(y+z)}{x(x+y+z)}\ge 0$$

or, equivalently,

$$\frac{x(z+t)+y^2}{y(z+t)+y^2}+\frac{y(x+t)+z^2}{z(x+t)+z^2}+\frac{z(x+y)+t^2}{t(x+y)+t^2}+\frac{t(y+z)+x^2}{x(y+z)+x^2}\ge 4 .$$

Notice that by Cauchy-Bunyakovsky inequality $(x(z+t)+y^2)\left(\dfrac{z+t}{x}+1\right)\ge(z+t+y)^2$

or $\dfrac{x(z+t)+y^2}{y(z+t)+y^2}\ge\dfrac{x(z+t+y)}{y(z+t+x)}$.

(Equality occurs iff $x = y$.)

Now, writing similar inequalities for other terms we obtain

$$\frac{x(z+t)+y^2}{y(z+t)+y^2}+\frac{y(x+t)+z^2}{z(x+t)+z^2}+\frac{z(x+y)+t^2}{t(x+y)+t^2}+\frac{t(y+z)+x^2}{x(y+z)+x^2}\geq$$

$$\geq\frac{x(z+t+y)}{y(z+t+x)}+\frac{y(x+t+z)}{z(x+t+y)}+\frac{z(x+y+t)}{t(x+y+z)}+\frac{t(y+z+x)}{x(y+z+t)}\geq$$

$$\geq 4\sqrt[4]{\frac{x(z+t+y)}{y(z+t+x)}\frac{y(x+t+z)}{z(x+t+y)}\frac{z(x+y+t)}{t(x+y+z)}\frac{t(y+z+x)}{x(y+z+t)}}=4$$

Equality holds iff $x=y=z=t$ i.e. $a=b=c=d=1$.