THEORETICAL COMPETITION

January 16, 2011

Please read this first:

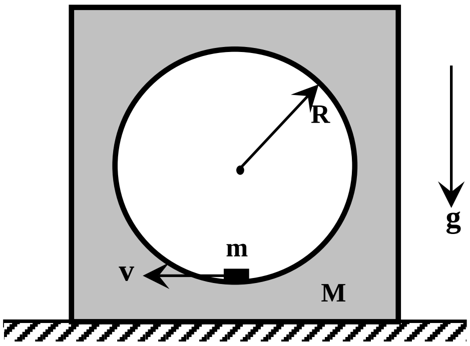
1. The time available for the theoretical competition is 4 hours. There are three questions.
2. Use only the pen provided.
3. You can use your own calculator for numerical calculations. If you don't have one, please ask for it from Olympiad organizers.
4. You are provided with ***Writing sheet*** and additional paper. You can use the additional paper for drafts of your solutions but these papers will not be checked. Your final solutions which will be evaluated should be on the ***Writing sheets***. Please use as little text as possible. You should mostly use equations, numbers, figures and plots.
5. Use only the front side of ***Writing sheets*.** Write only inside the boxed area.
6. Begin each question on a separate sheet of paper.
7. Fill the boxes at the top of each sheet of paper with your country (**Country),** your student code (**Student Code**), the question number (**Question Number)**, the progressive number of each sheet **(Page Number**), and the total number of ***Writing sheets*** used (**Total Number of Pages**). If you use some blank ***Writing sheets*** for notes that you do not wish to be evaluated, put a large X across the entire sheet and do not include it in your numbering.
8. At the end of the exam, arrange all sheets for each problem in the following order:
   - Used ***Writing sheets*** in order;
   - The sheets you do not wish to be evaluated
   - Unused sheets and the printed question.

   Place the papers inside the envelope and leave everything on your desk. You are not allowed to take any paper out of the room.

Problem 1 (10 points)

This problem consists of three independent parts.

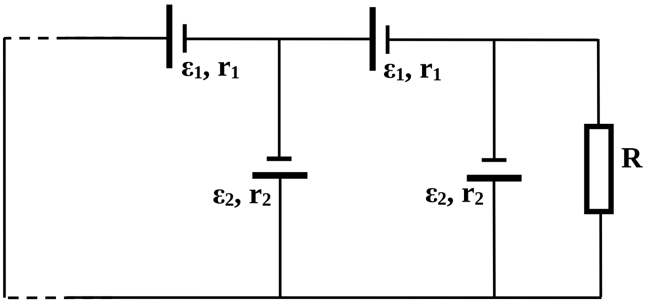
1A (3.5 points)



The body is a cube in the center of which a spherical cavity of radius R is carved out. Inside the spherical cavity at its bottom there is a motionless puck whose geometric sizes are negligible. Find the minimal horizontal velocity (at all possible cube-to-puck mass ratios) which has to be imposed on the puck so as the cubic body should jump up from the table surface in the motion process followed. Friction in the system is completely absent. At which value of the cube-to-puck mass ratio M/m that minimal velocity is achieved?

1B (4 points)

The resistance $R = 2.0\,\Omega$ is connected to an infinite number of current sources obtained by repetition as shown in the figure on the right. Find the electric current flowing through the resistance R. The emfs' and internal resistances of the sources are known to be $\varepsilon_1 = 2.0\,\text{V}$, $r_1 = 1.0\,\Omega$, and $\varepsilon_2 = 1.0\,\text{V}$, $r_2 = 2.0\,\Omega$.



1C (2.5 points)

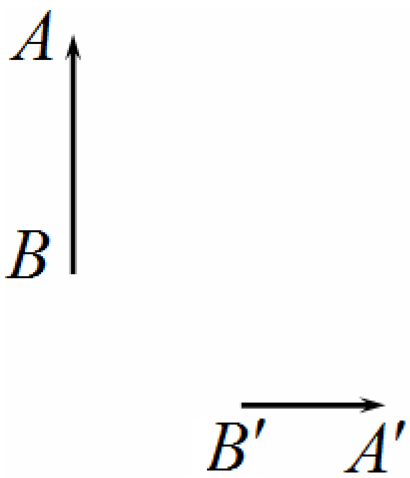
In Figure on the right b, you can see the object AB and its image A'B' in a thin lens. By drawing method, please find:

a) the optical center of the lens (0.5 points);

б) the lens plane (1 point);

в) the main focuses of the lens (0.5 points).

Is the lens concave (diverging) or convex (collecting)? Please write down the answer. (0.5 points).



Problem 2 (10 points)
Electrical conductivity of metals (10 points)

Ohm's law

Conductors are materials, usually metals, in which an ordered motion of free charges called an electric current is possible in the presence of an external electric field. The law relating the electric current strength I flowing through the conductor with a voltage U applied to its ends was experimentally discovered by Georg Ohm (1787-1854) and has the following form:

$$I = \frac{U}{R}, \tag{1}$$

where R is called the resistance of the conductor.

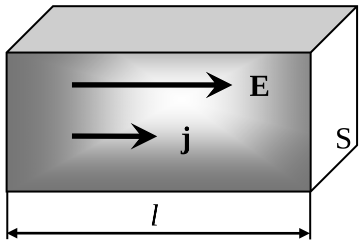


Fig.1

Consider a small element of the metallic material with the length l and the cross section S, whose ends are subject to the voltage U. Let σ be the specific electrical conductivity of the substance which is the quantity inverse to the specific electrical resistivity ρ. The resistance of the conductor and the electric current strength flowing through it are written as

$$R = \rho\frac{l}{S} = \frac{1}{\sigma}\frac{l}{S}, \qquad I = jS, \tag{2}$$

where the current density j is introduced, representing the amount of the electric charge that passes through the unit of the cross section in the unit of time. The current density depends on the electron number density and electron **average ordered velocity**.

Taking into account that $E = U/l$ stands for the electric field strength, the local (differential) form of Ohm's law is obtained from Eqs. (1) and (2) as

$$j = \sigma E. \tag{3}$$

Accounting for the same direction of the electric field strength and the current density vectors, relation (3) can be rewritten in vector form

$$\mathbf{j} = \sigma\mathbf{E}. \tag{4}$$

1. **[1 point]** Starting from the Joule-Lenz law, discovered first by James Joule and later by Heinrich Lenz, determine the volume density of the thermal power P_V released in the conductor, i.e. the amount of heat generated by the electric current in the unit volume 1 m^3 in the unit of time 1 s. Express your answer in terms of E and σ.

The Drude model

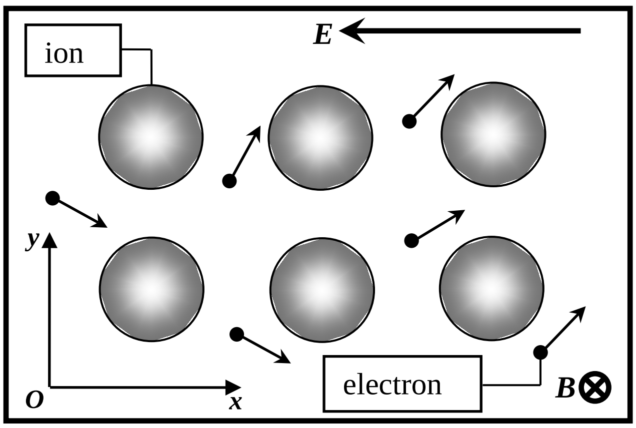


Fig.2

After the discovery of the electron in 1900 by Joseph John Thompson, German physicist Paul Drude proposed the so-called classical theory of electrical conductivity of metals. According to this theory, the electrons with the number density n, the mass m and the electric charge $-e$ can move freely in the ionic crystal lattice of metal, occasionally colliding with ions located at the sites and thereby transferring their kinetic energy to ions.

Real motion of an electron is very complicated because of the chaotic thermal motion. Under the influence of the external field, all electrons acquire the same acceleration and thus gain some extra velocity. This results in an ordered motion of electrons causing an electrical current to appear. We are only interested in that ordered motion of electrons which is superimposed on their random thermal walks.

Since the real picture of the electrical conductivity is very complicated, we adopt the following simplified model. Assume that an electron with an initial zero velocity accelerates over time interval τ and then collides with an ion at the site, thereby transferring all the acquired kinetic energy. Then it again starts to accelerate and over time interval τ collides with another ion and so on. Keeping this in mind find answers to the following: *In those processes the electrons do not interact with each other.*

2. **[1 point]** Determine the vector of average ordered velocity of electrons $\mathbf{u}$. Express your answer in terms of e, $\mathbf{E}$, m, and τ.

3. **[1 point]** The current density in the sample is determined by the average velocity component parallel to the external electric field strength E. Show that Ohm's law holds in this simplified model and determine the specific conductivity σ of the metal. Express your answer in terms of e, n, m, and τ.

4. **[1 point]** Determine the amount of the kinetic energy Q_V transferred by electrons to the crystal lattice in the unit volume 1 m^3 and in the unit of time 1 s. Express your answer in terms of e, $\mathbf{E}$, n, m, and τ.

Magnetoresistance

One of the important galvanomagnetic phenomenon is the change in the conductivity of a conductor that is subject to an external transverse magnetic field. This phenomenon is called a magnetoresistance effect. Due to experiments, the relative deviation of the specific conductivity $\Delta\sigma/\sigma$ at not very strong magnetic field with the induction B is given by the formula

$$\frac{\Delta\sigma}{\sigma} = \frac{\sigma(B)-\sigma(B=0)}{\sigma(B=0)} = \mu B^{\nu}, \tag{5}$$

where μ and ν are some constants.

Making use of the Drude model described above, solve the following problems. Carefully examine figure 2 presented above, since it represents the system of coordinates in use and displays directions of all vectors.

5. **[1 point]** Find the dependences on time t of the projections $u_x(t)$ and $u_y(t)$ of the electron velocity between two consecutive collisions. Express your answer in terms of e, E, B, m, and t.

6. **[2 points]** The current density in the sample is determined by the average velocity component parallel to the external electric field strength E. Assuming that the magnetic field induction B is small enough, find the constants μ and ν in formula (5). Express your answer in terms of e, m, and τ.

The Hall effect

In 1879, Edwin Hall discovered the phenomenon of appearance of the transverse potential difference, later called the Hall voltage, by placing the current-carrying conductor in a constant transverse magnetic field.

In the simplest consideration, the Hall effect is described as follows. Suppose that an electric current flows through a metal bar due to the applied external electric field of the strength E and is placed in a weak transverse magnetic field of induction B. The magnetic field deflects electrons from their straight-line motion to one of the bar faces. Thus, the Lorentz force, in contrast to the magnetoresistance effect, causes the accumulation of the negative charge near one of the bar faces and the positive charge near the opposite face. The accumulation of charge persists until the transverse electric field E_H, generated by the accumulated charges themselves (directed along the axis *Oy* as shown in the figure above), **does totally compensate** for the transverse displacement of electrons over the time interval τ.

Making use of the Drude model, described above, solve the following problems. Carefully examine figure 2 presented above, since it represents the system of coordinates in use and displays directions of all vectors.

7. **[0.5 points]** Look carefully at the second figure above. Near which of the faces, the top or the bottom, is the negative charge accumulated?

8. **[1.5 points]** Find the dependences on time t of the projections $u_x(t)$ and $u_y(t)$ of the electron velocity between two consecutive collisions. Express your answer in terms of e, E, E_H, B, m, and t.

9. **[1 point]** Find the Hall electric field strength E_H. Express your answer in terms of e, E, B, m, and τ, and then in terms of e, j, B, and n.

In solving these problems you can use the following approximate formulae valid for small values of x:

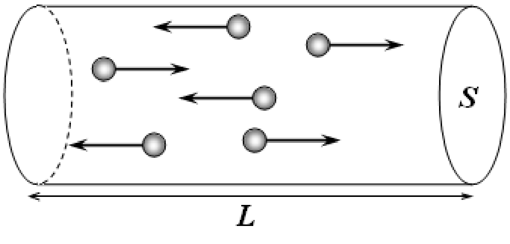
$$\sin x \approx x - \frac{x^3}{6}$$
$$\cos x \approx 1 - \frac{x^2}{2} + \frac{x^4}{24}.$$

Problem 3 (10 points)

Thermodynamics of simple quantum ideal gas

In classical physics, the energy of the system varies continuously. In the physics of the microworld, most physical parameters are quantized, i.e., they can take a discrete set of values. Quantization of energy can result in actually observed macroscopic effects. In this problem, you are to consider the simplest model of a quantum ideal gas.

Model



Let a gas consist of *N* identical atoms of mass *m*, which are placed in a long cylindrical vessel of length *L* and cross-section *S*. Atoms can only move along the axis of the vessel. The kinetic energy of atoms is quantized, i.e. it can take a discrete set of values determined by

$$E_n = n\varepsilon, \tag{1}$$

where $n = 1, 2, 3...$ and ε is a known constant value. Assume that the kinetic energy of an atom is expressed by the formula of classical physics.

The vessel is brought into contact with the thermostat so that the gas temperature in the vessel is T. The value of kinetic energy of a single atom is changed due to contact with the thermostat. Assume that the atom number density is such low that collisions between atoms are rare and can be neglected.

At thermodynamic equilibrium the number of atoms, which occupy the level with the energy E_n, is determined by the Boltzmann distribution function of the form

$$N_n = C\exp\left(-n\frac{\varepsilon}{k_B T}\right), \quad (2)$$

where k_B denotes the Boltzmann constant, C is a normalization factor that you have to determine by yourself.

Subproblems:

1 **[1 point]** Find the number of atoms N_n that occupy the energy level E_n. Express your answer in terms of N, ε, T, and k_B.

2 **[3 points]** Find the expression for the internal energy U of the gas. Express your answer in terms of N, ε, T, and k_B. Obtain approximate formulae for the internal energy of the gas in two limiting cases: $k_B T >> \varepsilon$ (**high temperature limit or classical limit**) and $k_B T << \varepsilon$ (**low temperature limit**).

3 **[3 points]** Calculate the molar heat capacity of gas at constant volume. Express your answer in terms of N, ε, T, and k_B. Obtain approximate formulae for the molar heat capacity at constant volume both in the classical limit and the limit of low temperatures. Draw a schematic plot of the molar heat capacity dependence against the gas temperature.

4 **[3 points]** Find the pressure P exerted by the gas on the vessel walls. Express your answer in terms of N, ε, T, and k_B. Obtain approximate formulae for the pressure both in the classical limit and the limit of low temperatures. Draw a schematic plot of the pressure dependence against the gas temperature.

In solving these problems you can use the following formulae:

$$\sum_{n=1}^{\infty} x^n = \frac{x}{1-x},$$

$$\sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2},$$

$$\exp(x) \approx 1 + x, \quad x<<1,$$

$$\frac{1}{1-x} \approx 1 + x, \quad |x| << 1.$$