4. Find all $k > 0$ for which a strictly decreasing function $g : (0, +\infty) \to (0, +\infty)$ exists such that $g(x) \geqslant kg(x + g(x))$ for all positive x.

The answer is $k \leqslant 1$.

Solution. Every $k \leqslant 1$ satisfies the condition because for any decreasing g, for instance, $g(x) = \frac{1}{x}$, the inequality $g(x) > g(x + g(x)) \geqslant kg(x + g(x))$ holds.

Suppose a function g satisfies the condition for some $k > 1$. Let $s = \frac{1}{k}$, then $g(x + g(x)) \leqslant sg(x)$.

Define the sequence (x_n) by $x_0 = x$, $x_{n+1} = x_n + g(x_n)$. We have $g(x_{n+1}) \leqslant sg(x_n)$, therefore $g(x_n) \leqslant s^n g(x)$. Since

$$x_n = x_0 + g(x_0) + g(x_1) + \ldots + g(x_{n-1}) \leqslant x + g(x) + sg(x) + \ldots + s^{n-1}g(x) =$$

$$= x + (1 + s + \ldots + s^{n-1})g(x) < x + \frac{1}{1-s}g(x),$$

it follows that $g(x_n) > g(x + \frac{1}{1-s}g(x))$. Thus $g(x + \frac{1}{1-s}g(x)) < s^n g(x)$ for every positive integral n, which is clearly impossible, because $g(x + \frac{1}{1-s}g(x)) > 0$. The contradiction proves that the case $k > 1$ is impossible.

5. A convex hexagon $ABCDEF$ is given such that $AB \parallel DE$, $BC \parallel EF$, $CD \parallel FA$. The points M, N, K are common points of the lines BD and AE, AC and DF, CE and BF respectively. Prove that the perpendiculars drawn from M, N, K to lines AB, CD, EF respectively are concurrent.

Solution. We make use of the following

Lemma. Let T be the common point of extended legs PS and QR of a trapezium $PQRS$. Then the radical axis of two circles with diameters PR и QS is the altitude of the triangle ABC drawn from T.

Proof. We consider three circles: ω_1 with diameter PR, ω_2 with diameter QS, and ω with diameter PQ. The common chord of ω and ω_1 is the altitude drawn from P to QR, and the common chord of ω and ω_2 is the altitude drawn from Q to PR. The common point of these altitudes, that is, the orthocentre of TPQ, has equal degrees with respect to ω_1 and ω_2, and therefore belongs to their radical axis. Similarly, their radical axis contains the orthocentre of TRS. Since the perpendicular drawn from T to PQ contains both the orthocentres, it is the radical axis of ω_1 and ω_2.

Applying the lemma to the trapezium $ABDE$, we find that the perpendicular drawn from M to AB is the radical axis of the circles with diameters AD and BE. Considering the trapezia $CDFA$ and $EFBC$ in the same way, we prove that the perpendicular drawn from N to CD is the radical axis of the circles with diameters AD and CF, and the perpendicular drawn from K to EF is the radical axis of the circles with diameters CF and BE. It follows that the three perpendiculars (no two of which are parallel) have a common point: the radical centre of the three circles.

6. We call a positive integer q *a convenient denominator* for a real number α if $|\alpha - \frac{p}{q}| < \frac{1}{10q}$ for some integer p. Prove that if two irrational numbers α and β have the same set of convenient denominators then either $\alpha + \beta$ or $\alpha - \beta$ is an integer.

Solution.

Let $q_1 < q_2 < \ldots$ be all convenient denominators for the numbers α and β. For each q_i obviously there exists a unique p_i such that $|q_i\alpha - p_i| < \frac{1}{10}$; we call this p_i *convenient numerator* corresponding to q_i.

First we consider the case when $0 < \alpha, \beta < \frac{1}{10}$. Let $p_1 < p_2 < \ldots$ be all the convenient numerators for α and $p'_1 < p'_2 < \ldots$ all the convenient numerators for β. We prove by induction in i that $p_i = p'_i$ for all i. Clearly $p_1 = p'_1 = 0$. Suppose $p_k = p'_k$. If $q_{k+1} = q_k + 1$ then $p_{k+1} = p_k$ (since $|p_k - q_k\alpha| < \frac{1}{10}$ and $|p_{k+1} - q_{k+1}\alpha| = |p_k - q_k\alpha - \alpha| < \frac{1}{10}$); similarly, $p'_{k+1} = p'_k$, therefore $p_{k+1} = p'_{k+1}$. If $q_{k+1} > q_k + 1$ then $p_{k+1} = p_k + 1$. Indeed, the first term of the arithmetical progression with difference $\alpha < \frac{1}{10}$ starting from $(q_k + 1)\alpha$ is less than $(p_k + 1) - \frac{1}{10}$; it follows that there must be a term differing from $p_k + 1$ by less than $\frac{1}{10}$. Similarly, $p'_{k+1} = p'_k + 1$, and the statement is proved.

Since $|q_k\alpha - p_k| < \frac{1}{10}$ and $|q_k\beta - p_k| < \frac{1}{10}$, we get $|q_k(\alpha - \beta)| < \frac{1}{5}$ for all k, hence $\alpha = \beta$.

In the case of arbitrary α and β we consider the numbers $q_1\alpha$ and $q_1\beta$. Changing signs if necessary, we may assume that $0 < \{q_1\alpha\}, \{q_1\beta\} < \frac{1}{10}$. Since the numbers $q_1\alpha$ and $q_1\beta$ satisfy the original condition, $\{q_1\alpha\}$ and $\{q_1\beta\}$ also satisfy it, therefore $\{q_1\alpha\} = \{q_1\beta\}$. This means that $q_1\alpha - q_1\beta = r$ is an integer, i.e. the difference $\alpha - \beta = \frac{r}{q_1}$ is rational.

Suppose $\frac{r}{q_1}$ is not an integer. Then $\frac{1}{3} \leqslant \{\frac{kr}{q_1}\} \leqslant \frac{2}{3}$ for some k. We shall use the fact that for each u and v, $0 \leqslant u < v \leqslant 1$, every arithmetical progression with irrational difference α contains a term with fractional part in the interval (u, v). In particular, for some positive integer n the difference between the number $(nq_1 + k)\alpha$ and the nearest integer is less than $\frac{1}{10}$. But then this must be true for the number $(nq_1 + k)\beta$, which is impossible because $\{(nq_1 + k)\alpha - (nq_1 + k)\beta\} = \{nr + \frac{kr}{q_1}\} \in [\frac{1}{3}, \frac{2}{3}]$.