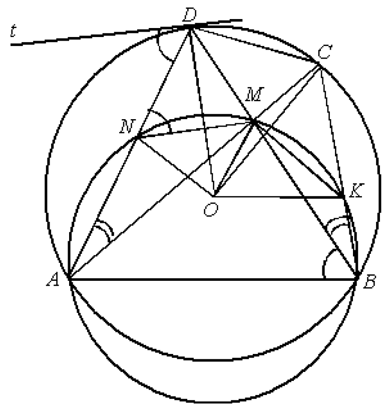
1. A quadrilateral $ABCD$ is inscribed in a circle with centre O. Its diagonals meet at M. The circumcircle of ABM intersects the sides AD and BC at N и K respectively. Prove that the areas of $NOMD$ and $KOMC$ are equal.

Solution. Let ω_1 be the circumcircle of $ABCD$ and ω_2 the circumcircle of ABM. The angles $\angle CAD$ and $\angle DBC$ are subtended by the same arc of ω_1 and therefore equal. It follows that the chords MN and MK of ω_2 subtending these angles are equal. The segments OD and OC are equal as radii of ω_1. Let t be the tangent to ω_1 at D. The angle between t and AD is equal to $\angle ABD$ (since it equals half of the arc AD) and therefore to $\angle MND$ (because the quadrilateral $ABMN$ is cyclic). Thus MN is parallel to t and consequently perpendicular to OD. Similarly, MK is perpendicular to OC. It follows now that the quadrilaterals $NOMD$ and $KOMC$ have equal area since their respective diagonals are equal and in both quadrilaterals the diagonals are perpendicular.



2. The numbers a_1, a_2, …, a_{100} are a permutation of the numbers 1, 2, …, 100. Let $S_1 = a_1$, $S_2 = a_1 + a_2$, …, $S_{100} = a_1 + a_2 + \ldots + a_{100}$. What maximum number of perfect squares can be among the numbers S_1, S_2, …, S_{100}?

The answer is 60.

Solution. We add initial term $S_0 = 0$ to the sequence S_1, S_2, …, S_{100} and consider all the terms $S_{n_0} < S_{n_1} < \ldots$ that are perfect squares: $S_{n_k} = m_k^2$ (in particular, $n_0 = m_0 = 0$). Since $S_{100} = 5050 < 72^2$, all the numbers m_k do not exceed 71. If $m_{k+1} = m_k + 1$ the difference $S_{n_{k+1}} - S_{n_k} = 2m_k + 1$ is odd, and an odd number must occur among the numbers a_{n_k+1}, …, $a_{n_{k+1}}$. There are only 50 odd numbers less than 100, so at most 50 differences $m_{k+1} - m_k$ equal 1. If there is 61 perfect squares in the original sequence, then $m_{61} = (m_{61} - m_{60}) + (m_{60} - m_{59}) + \ldots + (m_1 - m_0) \geqslant 50 + 11 \cdot 2 = 72$, a contradiction.

It remains to give an example of sequence containing 60 perfect squares. Let $a_i = 2i - 1$ for $1 \leqslant i \leqslant 50$, then we use all the odd numbers and $S_i = i^2$. Further, let $a_{51+4i} = 2 + 8i$, $a_{52+4i} = 100 - 4i$, $a_{53+4i} = 4 + 8i$, $a_{54+4i} = 98 - 4i$ for $0 \leqslant i \leqslant 7$; thus we use all the even numbers between 70 and 100 and all the numbers between 2 and 60 that leave the remainder 2 or 4 when divided by 8. For $0 \leqslant i \leqslant 7$ we have $S_{54+4i} - S_{50+4i} = 204 + 8i$, and $S_{54+4i} = (52 + 2i)^2$. Finally, let the last 18 terms of the sequence be 30, 40, 64, 66, 68, 6, 8, 14, 16, 32, 38, 46, 54, 62, 22, 24, 48, 56. This gives $S_{87} = 66^2 + 2 \cdot 134 = 68^2$, $S_{96} = 70^2$.

3. There are 60 towns in Graphland; every two towns are connected with a one-way road. Prove that one can colour four towns red and another four towns green so that every road between a red town and a green town is directed from the red town to the green one.

Solution. We say that town A *serves* a quadruple of towns B_1, B_2, B_3, B_4 if roads go from it to all these four towns. If k roads go from a town, the town serves $\binom{k}{4}$ quadruples. Let the numbers of roads going from all the towns be k_1, k_2, …, k_{60}. The sum of these numbers is the total number of roads, i.e. $\binom{60}{2} = 30 \cdot 59$. The total number of quadruples served by all the towns is $S = \binom{k_1}{4} + \binom{k_2}{4} + \ldots + \binom{k_{60}}{4}$. We claim that tne minimum of this sum on condition that $k_1 + k_2 + \ldots + k_{60} = 30 \cdot 59$ is $30 \cdot \binom{30}{4} + 30 \cdot \binom{29}{4}$. Indeed, the number of sets $(k_1, \ldots, k_{60})$ with sum $30 \cdot 59$ is finite, therefore on of these sets gives the sum its minimum value. Suppose this set contains two numbers $m \geqslant 4$ and n such that $m - n \geqslant 2$. Then replacing m и n by $m-1$ and $n+1$ decreases our sum (since $\binom{m}{4} + \binom{n}{4} - \binom{m-1}{4} - \binom{n+1}{4} = \binom{m-1}{3} - \binom{n}{3} > 0$). Thus the minimum value of S is attained for the set of k_i where the difference of every two numbers does not exceed 1. Such set is obviously unique and consists of 30 numbers equal to 30 and 30 numbers equal to 29.

Now the total number of quadruples served by all 60 towns is at least $30 \cdot \binom{30}{4} + 30 \cdot \binom{29}{4}$. It is easy to check however that this number is greater than $3 \cdot \binom{60}{4}$, that is, thrice the number of all quadruples. Therefore there is a quadruple served by four towns, q.e.d.