Solutions for 1-st round IZhO 2018

1. Let α, β, γ be the angles of a triangle opposite to the sides a, b, c respectively. Prove the inequality

$$2\left(\cos^2\alpha+\cos^2\beta+\cos^2\gamma\right)\geq\frac{a^2}{b^2+c^2}+\frac{b^2}{a^2+c^2}+\frac{c^2}{a^2+b^2}.$$

Solution. By the Law of Sines, RHS equals $\frac{\sin^2\alpha}{\sin^2\beta+\sin^2\gamma}+\frac{\sin^2\beta}{\sin^2\alpha+\sin^2\gamma}+\frac{\sin^2\gamma}{\sin^2\alpha+\sin^2\beta}$.
Applying Cauchy-Bunyakowski inequality we have

$$\sin^2\alpha=\sin^2(\beta+\gamma)=(\sin\beta\cos\gamma+\sin\gamma\cos\beta)^2\leq(\sin^2\beta+\sin^2\gamma)(\cos^2\gamma+\cos^2\beta),$$

therefore $\cos^2\beta+\cos^2\gamma\geq\frac{\sin^2\alpha}{\sin^2\beta+\sin^2\gamma}$.
Adding similar inequalities for $\cos^2\gamma+\cos^2\alpha$ and $\cos^2\alpha+\cos^2\beta$ we get the desired result.

2. Points N, K, L lie on the sides AB, BC, CA of a triangle ABC respectively so that $AL=BK$ and CN is the bisector of the angle C. The segments AK and BL meet at the point P. Let I and J be the incentres of the triangles APL and BPK respectively. The lines CN and IJ meet at point Q. Prove that $IP=JQ$.

Solution. The case $CA=CB$ is trivial. If $CA\neq CB$, we may suppose, without loss of generality, that CN meets the segment PK.

Let the circumcircles ω_1 and ω_2 of the triangles APL and BPK respectively meet again at point T. Then

$$\angle LAT=\angle TPB=\angle TKB. \qquad (1)$$

and $\angle ALT=\angle APT=\angle TBK$, that is, $\triangle ALT=\triangle KBT$, hence

$$AT=TK. \qquad (2)$$

It follows from (1) that the quadrilateral $ACKT$ is cyclic; together with (2) this means that $\angle ACT=\angle TCK$, i.e. T lies on the bisector of CN.

Let IJ meet ω_1 and ω_2 at I_1 and J_1 respectively. Since ω_1 and ω_2 have equal radii and $AL=BK$, the triangles ALI_1 and BKJ_1 are equal. We use Mansion's lemma: the midpoint of arc XY of the circumcircle of XYZ lies at equal distances from the ends of this arc and the incentre. It follows from this lemma that $I_1I=I_1L=J_1K=J_1J$. Moreover, $\angle PI_1T=\angle PAT=\angle PKT=\angle PJ_1T$, therefore, $I_1T=J_1T$. Thus T lies on the median bisector of I_1J_1 and on the median bisector of IJ.

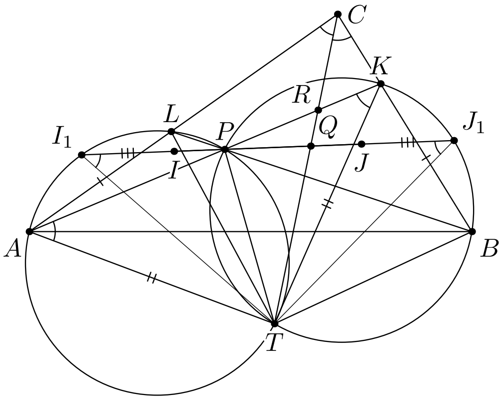

Figure 1: image

It remains to prove that T lies on the median bisector of PQ. Let $R=AK\cap CT$. Then $\angle ART=\angle RAC+\angle ACR=\angle RAC+\angle AKT=\angle RAC+\angle KAT=\angle LAT=\angle BPT$. Since PQ bisects the angle RPB, $\angle PQT=\angle PRT+\angle RPQ=\angle PBT+\angle BPJ=\angle TPQ$, therefore T belongs to the median bisector of PQ and $IP=JQ$.

3. Prove that there exist infinitely many pairs (m,n) of positive integers such that $m+n$ divides $(m!)^n+(n!)^m+1$.
Solution. We shall find a pair such that $m+n=p$ is prime and n is even. Applying Wilson's theorem we have

$$m!=(p-n)!=\frac{(p-1)!}{(p-n+1)\ldots(p-2)(p-1)}\equiv\frac{-1}{-(n-1)\ldots(-2)(-1)}\equiv\frac{1}{(n-1)!}\equiv\frac{n}{n!}\pmod p.$$

It follows from Fermat's Little Theorem that $(n!)^p\equiv n!\pmod p$, therefore

$$(m!)^n+(n!)^m+1\equiv\left(\frac{n}{n!}\right)^n+(n!)^{p-n}+1\equiv\frac{n^n+n!+(n!)^n}{(n!)^n}\pmod p;$$

thus it suffices to prove that the number $n^n+n!+(n!)^n$ has a prime divisor $p>n$ for infinitely many even n.

We prove that this condition is satisfied, for instance, by all the numbers of the form $n=2q$, where $q>2$ is prime. Let $A=(2q)^{2q}+(2q)!+((2q)!)^{2q}$. For a prime p and integer k we denote by $v_p(k)$ the largest integer ℓ such that p^ℓ divides k.

If $r<2q$ is prime and $r\notin\{2,q\}$ then $A\equiv(2q)^{2q}\not\equiv 0\pmod r$. The largest degree of q dividing $(2q)!$ is q^2, while for $(2q)^{2q}$ and $((2q)!)^{2q}$ it is $2q$ and $4q$ respectively, therefore $v_q(A)=2$.

Finally, $v_2((2q)!)=\left[\frac{2q}{2}\right]+\left[\frac{2q}{4}\right]+\left[\frac{2q}{8}\right]+\cdots<\frac{2q}{2}+\frac{2q}{4}+\frac{2q}{8}+\cdots=2q$, so $v_2((2q)!)<v_2((2q)^{2q})$ and obviously $v_2((2q)!)<v_2((2q)!^{2q})$, thus $v_2(A)\leq 2q-1$. On the other hand, $A>(2q)^{2q}>2^{2q-1}q^2$, therefore A has a prime divisor $p>2q$, q.e.d.