THEORETICAL COMPETITION

January 14, 2017

Please read this first:

1. The time available for the theoretical competition is 4 hours. There are three questions.
2. Use only the pen provided.
3. You can use your own calculator for numerical calculations. If you don't have one, please ask for it from Olympiad organizers.
4. You are provided with ***Writing sheet*** and additional paper. You can use the additional paper for drafts of your solutions but these papers will not be checked. Your final solutions which will be evaluated should be on the ***Writing sheets***. Please use as little text as possible. You should mostly use equations, numbers, figures and plots.
5. Use only the front side of ***Writing sheets.*** Write only inside the boxed area.
6. Begin each question on a separate sheet of paper.
7. Fill the boxes at the top of each sheet of paper with your country (**Country),** your student code (**Student Code**), the question number (**Question Number)**, the progressive number of each sheet **(Page Number**), and the total number of ***Writing sheets*** used (**Total Number of Pages**). If you use some blank ***Writing sheets*** for notes that you do not wish to be evaluated, put a large X across the entire sheet and do not include it in your numbering.
8. At the end of the exam, arrange all sheets for each problem in the following order:
   - Used ***Writing sheets*** in order;
   - The sheets you do not wish to be evaluated
   - Unused sheets and the printed question.

   Place the papers inside the envelope and leave everything on your desk. You are not allowed to take any paper out of the room.

Problem 1 (10.0 points)

This problem consists of three independent parts.

Problem 1A (3.0 points)

A new rapid-firing multibarrel machine gun is tested to provide $n = 100$ shots/s. The flight speed of a bullet is $u = 1000$ m/s, and its mass is equal to $m = 10$ g. The target is a sandbox of mass $M = 1000$ kg vertically suspended on a rope. Considering that the bullets are all stuck in the sandbox, evaluate the maximum deflection angle of the sandbox from the vertical after the shooting has started.

Problem 1B (4.0 points)

There is a bubble of radius R_1 somewhere in free space. With the help of an external ionizer the soap film is quickly charged up to some positive value, then over certain period of time the radius of the bubble ceases to change and becomes equal to $R_2 = 2R_1$. Find the electric charge q, which has been acquired by the soap film, if its heat capacity and heat conductivity are both negligible. The surface tension σ of the soap film does not depend on the temperature. The air is considered as an ideal diatomic gas.

Problem 1C (3.0 points)

Two identical source of coherent monochromatic waves with the wavelength λ are placed at the points S_1 and S_2 (see the provided separate sheet of paper for this subproblem). On the sheet of paper provided one wavelength corresponds to the size of the two squares. Receivers of waves are located at the points A_1 and A_2. Each source emits waves with the same intensity I_0 such that the change in the wave amplitude can be neglected with the distance from the sources. On the same sheet of paper for this subproblem plot those points in the highlighted oval area at which a third source should be placed to completely suppress the signals at the points A_1 and A_2. simulateously. What should be the intensity of the wave from the third source? All sources emit waves with the same phase and polarization perpendicular to the plane of the figure.

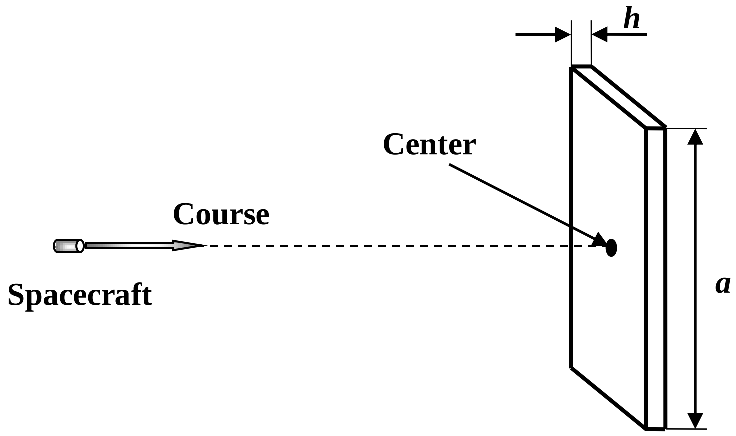
Attention! Make all the necessary drawings in the same sheet of paper provided for this subproblem, collect it together with the answer sheets **Writing sheets**, incorporating it into the overall numbering. Otherwise, your answer to this subproblem will not be evaluated!

Problem 2 (10.0 points)
Fantastic trip through the Universe

A reconnaissance spacecraft of the well developed civilization plows the universe. In this problem, you are asked to consider a few situations of that interstellar travel from the physical point of view. For all numerical calculations consider the gravitational constant known and equal to $G = 6.672 \times 10^{-11} m^3/(kg \cdot s^2)$.

1. Planets with strange shapes (3.9 points)

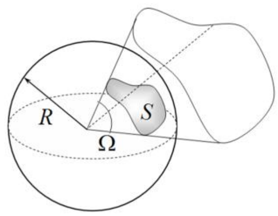
At some distance from the spacecraft the crew captain discovers the first planet that has a strange shape of a parallelepiped with a square base of side a and a very small thickness $h \ll a$. The captain gives the order to pursue a course to planet's center as shown in the figure below.



It has been revealed after turning off the engine that the spacecraft acceleration of the free fall g, that the planet provides at distances much greater h, remains proportional to the solid angle Ω, at which the planet is seen from the spacecraft, that is:

$$g = \alpha\Omega.$$

The solid angle Ω is a part of the space, which unites all the rays emanating from a given point (the vertex) and intersecting a surface (which is called a surface subtending the solid angle). The boundary of the solid angle is a certain conical surface.



The solid angle is measured by the ratio of the area S of the sphere centered at the vertex, which is cut by this solid angle, to the square of the sphere radius:

$$\Omega = \frac{S}{R^2}.$$

Solid angles are measured by abstract dimensionless quantities. The unit of the solid angle is a steradian, which, in SI units, is equal to the solid angle cutting out the surface area of R^2 from the sphere of radius R. The whole sphere corresponds to the solid angle of 4π steradian (full solid angle) from any vertex, situated inside the sphere, in particular, for the center of the sphere.

After landing on the planet surface and taking the soil samples, the scientists reported to the captain that the planet is composed of homogeneous material of the density $\rho_1 = 3000\, kg/m^3$ and the free fall acceleration near the geometric surface center of the planet remains almost constant and is equal to $g_1 = 9{,}81 \times 10^{-2} m/s^2$.

1.1 **[0.7 points]** Find and calculate the thickness of the planet h.
1.2 **[0.5 points]** Find and calculate the coefficient α.

After leaving the first planet, the captain and his crew meet another exotic planet that shapes a regular pyramid with a square base of the side $a = 10000km$ and of the height $\frac{a}{2}$.

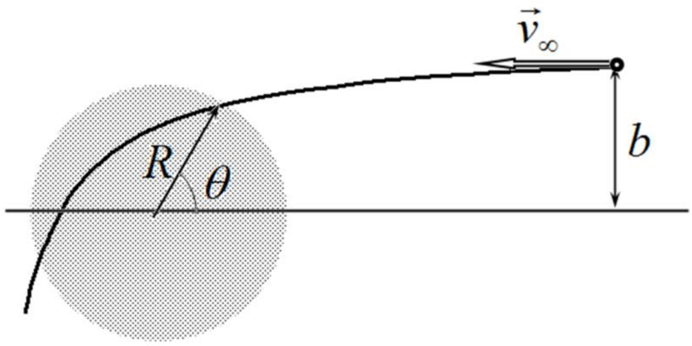
1.3 **[0.7 points]** Find and calculate the free fall acceleration g_2 measured at the top of the uniform pyramidal planet, if its density is equal to $\rho_2 = 4500kg/m^3$.

The spacecraft has left the pyramidal planet from its top, starting with the characteristic parabolic velocity $v_1 = 3{,}45km/s$. The next planet the spacecraft has met on its way is the one shaped like a perfect uniform cube with the side a. After accurate measurements, the captain and his crew have found that the density of the cubic planet is $\rho_3 = 5000kg/m^3$.

1.4 **[2.0 points]** Find and calculate the parabolic velocity v_2 for the spacecraft to start from one of the vertices of the cubic planet.

2. Dusty cloud (6.1 points)

The spacecraft encounters a very large massive dust cloud of radius $R = 1{,}50 \times 10^7 km$ and of homogeneous density $\rho_4 = 50{,}0kg/m^3$. The speed of the spacecraft at a large distance from the cloud reaches the value of $v_\infty = 100km/s$, and the impact parameter measured from the cloud center is equal to $b = 1{,}50 \times 10^8 km$. The engine remains switched off.



2.1 **[2.5 points]** Find and calculate the coordinate of the spacecraft entry into the dust cloud, characterized by the angle θ.

2.2 **[2.0 points]** Find and calculate the minimum distance r_{min}, the spacecraft flies by from the cloud center. Resistance to the motion of the spacecraft caused by the cloud particles can be neglected.

Making sure that it is impossible to avoid a collision with the cloud, the captain of the spacecraft takes the decision to turn on the engine, thereby increasing the speed v_∞.

2.3 **[1.0 points]** Find and calculate the minimum speed $v_{\infty,min}$, at which the spacecraft passes safely by the dust cloud.

Successfully passing the obstacle, the captain and his crew have discovered that the particles of the dust clouds contain valuable elements.

2.4 **[0.6 points]** Find the minimum work A, which must be performed in order to gradually bring all the dust particles onto a very remote processing plant.

Problem 3 (10.0 points)
Resistance of a prism

1. Mathematical introduction (3,0 points)

By definition, it is believed that terms of the numerical sequence obey the recurrence relation if each successive term is expressed through the previous ones. For example, for a well known geometric progression we have

$$x_k = \lambda x_{k-1}, \tag{1}$$

where $k = 1, 2, 3, \ldots$, λ stands for a fixed number and zeroth term of the numerical sequence has some value of A, i.e. $x_0 = A$.

1.1 **[0.2 points]** Obtain an explicit formula for an arbitrary term of the sequence x_k, i.e. express it through the successive number k, the initial value A and λ.

Let us consider the number $\lambda = 2 + \sqrt{3}$. Taking its natural power of k, the result can be presented in the following form

$$\lambda^k = p_k + q_k\sqrt{3}, \tag{2}$$

where p_k, q_k denote some integer numbers.

1.2 **[0.4 points]** Find the recurrence relations, expressing the values of p_k, q_k through the previous values p_{k-1}, q_{k-1}. Find also the inverse relations, expressing p_{k-1}, q_{k-1} through p_k, q_k.

1.3 **[0.7 points]** Calculate the numerical values of the coefficients p_k, q_k for $k = 1, 2, 3, 4, 5$.

1.4 **[0.2 points]** Express the number $\lambda^{-k} = \left(2 + \sqrt{3}\right)^{-k}$ in terms of p_k, q_k.

Let the terms of a certain numerical sequence obey the recurrence relation

$$x_{k+1} = 4x_k - x_{k-1}, \quad k = 1, \ldots, N-1, \tag{3}$$

where it is known that N is some integer number, $x_0 = A$ and $x_N = B$, A, B designate some values.

1.5 **[1.0 points]** Obtain an explicit formula for an arbitrary term x_k of the sequence (3), i.e. express it in terms of the number k and values A, B, N.

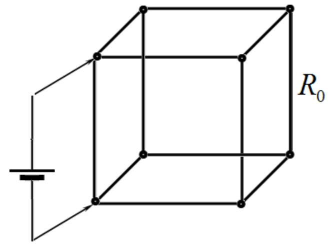
1.6 **[0.5 points]** Obtain an explicit formula for an arbitrary term x_k of the sequence (3) through p_k, q_k, found in.1.2-1.3.

Hint. The solution to the recurrence relation (3) must be sought in the form $x_k = C\lambda^k$, where C is a constant. Determine at what values of λ it is possible and construct an exact solution that satisfies all above stated conditions.

2. Wire frame in the shape of a prism (7.0 points)

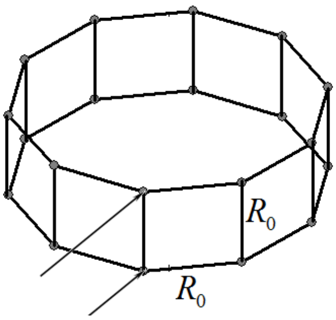
Problems are widely known in which you are asked to find the electrical resistance of a simple wire frame. An example of such a frame shaped in the form of a cube is shown below. Let the electric resistance of each edge be equal to R_0.

2.1 **[0.8 points]** Find the total resistance of the cube when the source is connected to the two adjacent cube vertices as shown on the right.

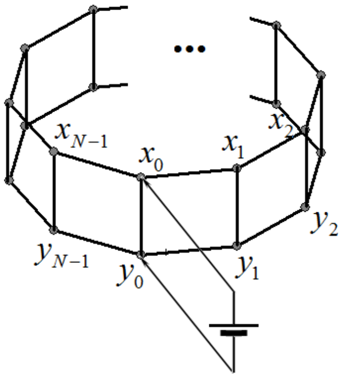


Let us now consider a more general case of the wire frame in the form of the regular prism with an arbitrary number N of side faces and determine its electrical resistance when the source is connected to the adjacent vertices of the side edge, as shown in the figure below. The resistance of each frame edge is equal to R_0.

For convenience, the vertices of the prism and their electric potentials on the upper and the bottom sides are consequently numbered and denoted, as shown in the figure below. DC voltage source is applied to zeroth vertices such that the source sets the potentials of the vertices equal to $x_0 = +\varphi_0$ and $y_0 = -\varphi_0$, respectively.



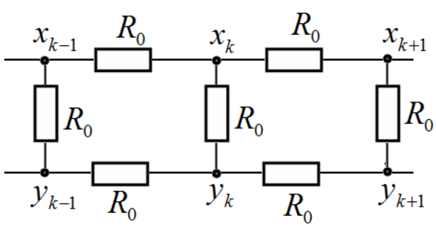
Wire frame in the shape of a prism



Numbering and denoting vertices' potentials

2.2 **[0.2 points]** Find the relation between the potentials x_k and y_k. Find the relation between the potentials x_k and x_{N-k}

Consider an arbitrary lateral edge, except zeroth ($k = 0$) and the last ($k = N-1$) ones. The corresponding circuit diagram is shown below.



2.3 **[1.0 points]** Find an recurrence relation for the potential x_k as expressed in terms of the neighboring vertices potentials for $k = 1,2...N-2$.

2.4 **[0.2 points]** Find boundary conditions at $k = 0$ and $k = N-1$ necessary for unambiguous determination of the potential x_k.

2.5 **[0.2 points]** Find explicit expressions for the potentials x_k and y_k for all possible numbers $k = 0,1,2...N-1$.

2.6 **[0.4 points]** Express the source current in terms of φ_0, R_0, N. Use the appropriate numbers p_k, q_k .obtained in the Mathematical introduction to this problem.

2.7 **[0.2 points]** Derive an explicit formula for the resistance R_N of the wire frame, expressed in terms of R_0, p_N, q_N.

2.8 **[1.0 points]** Find and tabulate the exact values of the frame resistances for $N = 1,2,3,4,5$.

2.9 **[0.5 points]** Draw the equivalent circuits for the exotic prisms with $N = 1$ and $N = 2$.

2.10 **[1.0 points]** Find the resistance R_∞ of the wire frame at $N \to \infty$.

2.11 **[1.5 points]** Find the minimum value of N at which the prism resistance deviation from R_∞ does not exceed 2%.