1. Each integral point of the plane is coloured white or blue. Prove that one can choose a colour so that for every positive integer n there is a triangle of area n with three vertices of the chosen colour.

Solution. If every two neighbouring points (that is, points at distance 1) have different colours then, in fact, we have a monochromatic lattice of $\sqrt{2} \times \sqrt{2}$ squares, where triangle with any integral area is easily found.

When this is not the case, we consider neighbouring points A and B ($AB = 1$) of the same colour (say white). To find a triangle of area n, we need a white point on the line parallel to AB at the distance $2n$. If there is such point for each n, we are done. Otherwise, we have a line ℓ with blue points only.

Consider a line ℓ_1 "next to ℓ", that is, the line parallel to ℓ at distance 1 from it. If it contains a blue point, we have a triangle with blue vertices of area $\frac{n}{2}$ for each positive integer n.

The only remaining case is that of line ℓ_1 containing only white points. Then we consider the line $\ell_2 \neq \ell$ at distance 1 from ℓ_1, and again, if there is a white point on ℓ_2, we are done. Now, if all points of ℓ_2 are blue, then for each n we have a triangle of area n with three blue vertices.

2. Inside the triangle ABC a point M is given. The line BM meets the side AC at N. The point K is symmetrical to M with respect to AC. The line BK meets AC at P. If $\angle AMP = \angle CMN$, prove that $\angle ABP = \angle CBN$.

Solution. Let D, E, F be the feet of perpendiculars to BP, MP, BM respectively drawn through A, and G, Q, H be the feet of perpendiculars to BP, MP, BM respectively drawn through C.

Note that $\triangle AFM \sim \triangle CQM$ and $\triangle AME \sim \triangle CMH$, therefore $\frac{AF}{CQ} = \frac{AM}{CM} = \frac{AE}{CH}$. By symmetry we have also $CQ = CG$, $AE = AD$ and $\angle FAD = \angle FBD = \angle GCH$, therefore $\frac{AF}{CG} = \frac{AD}{CH}$. It follows that $\triangle FAD \sim \triangle GCH$, thus $\angle AFD = \angle CGH$.

Now the points A, B, F, D are concyclic, therefore $\angle ABP = \angle AFD$, and similarly $\angle CBN = \angle CGH$. Combining that with the above, we have $\angle ABP = \angle CBN$.

Problem 3

Setting $x=1$, $y=0$ in the initial equation

$$f(x^3+y^3+xy)=x^2f(x)+y^2f(y)+f(xy) \tag{1}$$

gives $f(0)=0$.

Taking $y=0$ in (1) we obtain

$$f(x^3)=x^2f(x). \tag{2}$$

Substituting $y=-x$ into (1) leads to

$$f(-x^2)=x^2f(x)+x^2f(-x)+f(-x^2)\Rightarrow f(-x)=-f(x). \tag{3}$$

From (1) and (3) it follows that

$$\begin{aligned} &f(x^3+y^3+xy)+f(x^3-y^3-xy)\\ &\quad =x^2f(x)+y^2f(y)+f(xy)+x^2f(x)-y^2f(y)-f(xy)=2x^2f(x)=2f(x^3). \end{aligned} \tag{4}$$

Note that for any $a,\ b\in\mathbb{R}$ there exist $x,\ y\in\mathbb{R}$ such that

$$a=x^3+y^3+xy,\quad b=x^3-y^3-xy.$$

To this end, we take x, y that satisfy the equations

$$x^3=\frac{a+b}{2},\quad y^3+xy=\frac{a-b}{2}$$

(we see that functions in left hand sides of the equations have the ranges $\mathbb{R}$). Therefore, we can rewrite (4) in the form

$$f(a)+f(b)=2f\Big(\frac{a+b}{2}\Big),\quad a,\ b\in\mathbb{R}.$$

Thus, we have

$$f(0)+f(a+b)=2f\Big(\frac{a+b}{2}\Big)\Rightarrow f(a+b)=f(a)+f(b),\quad a,\ b\in\mathbb{R}.$$

Further, we change $x\to x+1$ in (2), denote $c=f(1)$, and from additivity of f obtain

$$\begin{aligned} f((x+1)^3)&=(x+1)^2f(x+1)\\ \Leftrightarrow f(x^3)+3f(x^2)+3f(x)+c&=(x^2+2x+1)(f(x)+c)\\ \Leftrightarrow 3f(x^2)&=(2x-2)f(x)+(x^2+2x)c \end{aligned} \tag{5}$$

Substituting $x\to -x$ in (5), we get

$$3f(x^2)=(-2x-2)f(-x)+(x^2-2x)c=(2x+2)f(x)+(x^2-2x)c \tag{6}$$

From the equality of right hand sides of (5) and (6) we obtain

$$f(x)=cx.$$

It is easy to verify that this function satisfies the given equation for all $c\in\mathbb{R}$.

Answer: $f(x)=cx$, $c\in\mathbb{R}$.