THEORETICAL COMPETITION

January 14, 2014

Please read this first:

1. The time available for the theoretical competition is 4 hours. There are three questions.
2. Use only the pen provided.
3. You can use your own calculator for numerical calculations. If you don't have one, please ask for it from Olympiad organizers.
4. You are provided with ***Writing sheet*** and additional paper. You can use the additional paper for drafts of your solutions but these papers will not be checked. Your final solutions which will be evaluated should be on the ***Writing sheets***. Please use as little text as possible. You should mostly use equations, numbers, figures and plots.
5. Use only the front side of ***Writing sheets.*** Write only inside the boxed area.
6. Begin each question on a separate sheet of paper.
7. Fill the boxes at the top of each sheet of paper with your country (**Country),** your student code (**Student Code**), the question number (**Question Number)**, the progressive number of each sheet **(Page Number**), and the total number of ***Writing sheets*** used (**Total Number of Pages**). If you use some blank ***Writing sheets*** for notes that you do not wish to be evaluated, put a large X across the entire sheet and do not include it in your numbering.
8. At the end of the exam, arrange all sheets for each problem in the following order:
    - Used ***Writing sheets*** in order;
    - The sheets you do not wish to be evaluated
    - Unused sheets and the printed question.

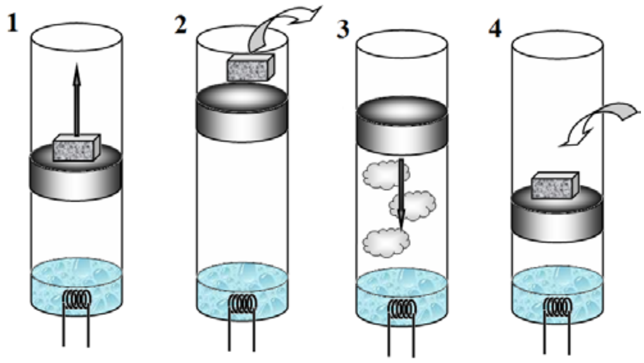
   Place the papers inside the envelope and leave everything on your desk. You are not allowed to take any paper out of the room.

Problem 1 (10 points)

This problem consists of three independent parts.

Problem 1A (3.0 points)

Steam engine consists of a vertical cylindrical vessel, in which a piston can move without friction. There is some water in the vessel with an electrical heater located inside. Steam engine cycle consists of four stages, as shown in the figure on the right:



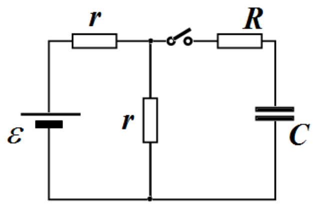
1. A load is placed on the piston, the heater is switched on making the water boiling and the vapor lifts up the piston with the load.
2. Once the piston has risen to some height, the load is quickly removed and the heater is immediately switched off.
3. The vapor under the piston cools down and condenses, the piston moves down slowly .
4. Once the piston has come down to some other height, the load is again put on the piston.

Draw a (P,V) diagram schematically showing the cycle of the steam engine and find its efficiency.

The atmospheric pressure is $P_0 = 1.0 \cdot 10^5 Pa$, the piston mass is $M = 2.0kg$, the piston area is $S = 10\,cм^2$, the load mass is $m = 1.0\,kg$, and the free fall acceleration is $g = 9.8m/s^2$. Assume that there is nothing under the piston except for the water vapor, and the dependence of the pressure of the saturated water vapor in the temperature range under consideration is approximated by the formula $P = at - b$, where $a = 4,85kPa/K$, $b = 384\,kPa$, t is the temperature in degrees Celsius.

Problem 1B (5.0 points)

In the circuit shown in the figure on the right, all the electrical components are ideal and their parameters are assumed to be given. Before switching on the key, the capacitor has been discharged. Find an amount of heat releasing in the resistor *R* after the key has been switched on.



Problem 1C (2.0 points)

Thin lens gives an image of the object, located perpendicular to its optical axis. The image size is 1 *cm*. If the distance from the object to the lens is increased by 5 *cm*, the image size remains 1 *cm*. Find the image size if the distance from the object to the lens is increased by another 5 *cm*.

Problem 2 Jet propulsion (10 points)

In a rocket engine thrust is created by the release of products of fuel combustion in the direction opposite to its motion. It is, of course, natural that the mass of the rocket decreases in the acceleration process. This idea was first proposed by the great Russian scientist K. Tsiolkovsky to implement the motion of objects in a vacuum, for example, in outer space. Nowadays space flights have become habitual. It is widely known that the space launching site, Baikonur, is situated on the territory of Kazakhstan. The first satellite and the first cosmonaut, Yu. Gagarin, were sent into space from Baikonur which is now a

complex of high-tech facilities intended to launch manned spacecraft into space, in particular, to the International Space Station.

Classical rocket

Let a rocket have an initial mass m_0 and let a fuel velocity relative to the rocket be constant and equal u. Assume that at the initial time moment the rocket is at rest in the laboratory frame of reference and no external force is present.

1. **[0.5 points]** Find the rocket velocity v as a function of its mass m. This formula is called after K. Tsiolkovsky. Express your answer in terms of m, m_0, u.

2. **[0.5 points]** An object of mass $m = 1000\,kg$ is required to be accelerated to the orbital velocity. Evaluate the initial rocket mass m_0, if the free fall acceleration is $g = 9.80\,m/c^2$ and the radius of the Earth is $R = 6400\,km$ and $u = 5,00\,km/s$.

Let a rocket move in the gravitational field of the Earth. The free fall acceleration g is assumed to be constant, whereas the fuel consumption $\mu(t) = -dm(t)/dt$ may depend on time.

3. **[0.75 points]** Write down the equation of motion of a rocket in Earth's gravitational field. This equation is called after I. Meshcherskij. Express your answer in terms of m, v, u, g, μ.

Assume in the following that the fuel exhaust velocity u is directed parallel to the free fall acceleration g, and the initial velocity of the rocket is zero.

4. **[0.75 points]** Find how the fuel consumption $\mu_{st}(t)$ should depend on time t in order for the rocket to hung motionless at some height. Express your answer in terms of m_0, u, g, t.

Assume now that the fuel consumption μ is also constant over time such that $\mu > \mu_{st}(t)$.

5. **[2.0 points]** In this case the rocket velocity dependence on time t can be represented as

$$\mathrm{v}(t) = A_1 t + A_2 \ln(1 + A_3 t),$$

where A_1, A_2, A_3 are some constants.

Find A_1, A_2, A_3 and express them in terms of m_0, u, g, μ.

6. **[1.0 points]** Suppose that the initial mass of the rocket is equal m_0, and the final mass is to be m. Find the maximum height $H_{\max}$ that the rocket can reach and determine the corresponding optimum fuel consumption μ_{opt}. Express your answer in terms of m_0, m, u, g.

Relativistic rocket

In the previous part of the problem it has been assumed that the rocket moves with a nonrelativistic velocity. To implement interstellar travels it is necessary to accelerate the rocket to the speed close to that of light and, then, relativity effects cannot be ignored at the calculations.

To establish the characteristic features of the rocket motion in a relativistic case, we introduce the concept of the proper frame of reference. The proper frame of reference is an inertial frame of reference which moves with the speed of the rocket itself relative to the laboratory reference frame, i.e. it is the reference frame in which the rocket is at rest at any given time.

7. **[2.5 points]** Find the relation between the rocket acceleration in the proper reference frame a_p and its acceleration in the laboratory frame of reference a_r when the velocity of the rocket is v, and c stands for the speed of light. Express your answer in terms of a_p, a_r, v, c.

8. **[1.5 points]** Let the rocket be at rest at the initial time moment. Then, using the results of the previous question it can be shown that at any time moment the rocket mass in the proper reference frame is related to its speed in the laboratory reference frame as

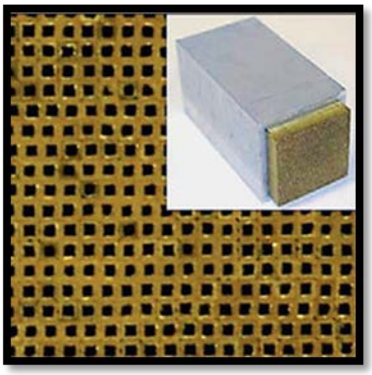
$$m = m_0 \left(\frac{1 - \mathrm{v}/c}{1 + \mathrm{v}/c}\right)^{\alpha}.$$

Find α and express it in terms of u, c.

9. **[0.25 points]** An object of mass $m = 1000\,kg$ is required to be accelerated to half the speed of light $\mathrm{v} = 0.5c$ where the speed of light is $c = 3.00 \cdot 10^8\,m/s$. Evaluate the initial rocket mass together with the fuel m_0 and **write it down as a power of 10**, if the fuel exhaust velocity is $u = 5,00\,km/s$.

10. **[0.25 points]** It can be shown that from the practical point of view the best rocket is the one that exploits photons rather than hot gases produced at the fuel combustion. An object of mass $m = 1000\,kg$ is required to be accelerated to half the speed of light $\mathrm{v} = 0.5c$. Evaluate the initial rocket mass together with the fuel m_0.

Problem 3 Metamaterials (10 points)



Metamaterials are composite materials whose properties are due not so much to the properties of its constituent elements but due to artificially tailored periodic structures. Metamaterials are synthesized in modern nanolaboratories by implantation different periodic structures with a variety of geometric shapes into the original natural material, which substantially modifies its physical properties. In a very rough approximation, those implants can be treated as artificially made atoms of extremely large size immersed into the original material. While synthesizing the metamaterial Ddveloper has the opportunity of varying various free parameters (structure sizes and constant or varying period between them, etc.).

In one nanolaboratory the metamaterial has been manufactured in the form of a wire of the length $L = 5.00cm$ and radius $R = 1.00mm$, whose conductivity depends on the distance from its axis according to the law $\sigma_0 = \beta r$. Physical properties of the wire have been experimentally measured and are presented in the following table:

PHYSICAL PROPERTY	NUMERICAL VALUE
Conductivity $\sigma_0 = \beta r$	$\beta = 1.00 \times 10^9\,S/m^2$
Heat transfer coefficient	$\alpha = 20W/(m^2 \cdot K)$
Thermal conductivity coefficient	$\kappa = 0,01W/(m \cdot K)$
Young's modulus	$E = 1.00 \times 10^7\,Pa$
Linear expansion coefficient	$\gamma = 1.00 \times 10^{-6}\,K^{-1}$

1. **[1.0 points]** Find an analytic formula for the total resistance R_0 of the wire, and calculate its numerical value.

An electric current $I = 1A$ is made to pass through the wire. It is known that the heat exchange with the environment obeys the Newton-Richman law,

$$P_{ext} = \alpha(T_s - T_0),$$

where P_{ext} stands for the power loss per unit surface of the wire with the surface temperature T_s, $T_0 = 293K$ denotes the ambient temperature and α is a constant, called the heat transfer coefficient.

2. **[1.0 points]** Find an analytic formula for the surface temperature T_s of the wire and calculate its numerical value.

The wire temperature varies with the depth due to the phenomenon known as thermal conductivity, which is described by the Fourier law

$$P = -\kappa S \frac{\Delta T}{\Delta x},$$

where P designates the power of the heat flow between the opposite faces of the parallelepiped with the square S, ΔT is the temperature difference between the faces of the parallelepiped situated at a distance Δx from each other, and κ is called the heat transfer coefficient.

3. **[2.5 points]** Find an analytic formula for the temperature $T_{\max}$ in the center of the wire, and calculate its numerical value.

4. **[0.5 points]** Find an analytic formula for the change δR_T of the wire radius due to its thermal expansion and calculate its numerical value.

Attention! *In all further calculations assume that the wire is infinitely long.*

5. **[0.5 points]** Find an analytic formula for the magnetic induction inside the wire as a function of the distance r from its axis.

6. **[1.0 points]** Find an analytic formula for the energy of the magnetic field inside the wire, and calculate its numerical value.

7. **[1.0 points]** The electric current causes an appearance of mechanical stress in the wire. Find an analytic formula for the pressure $p(r)$ inside the wire as a function of the distance r from its axis.

8. **[1.0 points]** Find an analytic formula for the mechanical stress energy W_σ of the wire, and calculate its numerical value.

9. **[1.0 points]** Find an analytic formula for the change δR_σ of the wire radius due to its mechanical stress, and calculate its numerical value.

10. **[0.5 points]** Find the value of the thermal expansion coefficient γ such that the total change of the wire radius would be zero when an electric current was passing through it.

Help! *The value of the magnetic constant is* $\mu_0 = 4\pi \cdot 10^{-7}\,Гн/м$.