4. Initial terms a_1, a_2, ..., a_k of a sequence (a_n) are different positive integers, and for $n > k$ the number a_n is the minimum positive integer not representable as a sum of some of the numbers a_1, a_2, ..., a_{n-1} (maybe one of them). Prove that $a_n = 2a_{n-1}$ for all large enough n.

Solution. For each $n \geqslant k$ we consider the set of all positive integers not exceeding $a_1 + a_2 + \ldots + a_n$; the numbers in this set that are not sums of elements of the set $\{a_1, a_2, \ldots, a_n\}$ we call *gaps*. If the set of gaps is not empty, then a_{n+1} is the minimum gap, otherwise $a_{n+1} = a_1 + a_2 + \ldots + a_n + 1$. We claim that when n increases the number of gaps decreases.

Note that if $t \leqslant S = a_1 + a_2 + \ldots + a_n$ is a sum of several numbers in the set $\{a_1, a_2, \ldots, a_n\}$, then $S - t$ is also such a sum: it is the sum of all a_i not included in the original sum.

The fact that a_{n+1} is the minimum gap means that all the numbers from 1 to $a_{n+1} - 1$ are sums of some of the numbers a_1, a_2, ..., a_n. Therefore all the numbers from $a_1 + \ldots + a_n - a_{n+1}$ to $a_1 + \ldots + a_n$ are such sums too. Adding a_{n+1} we see that all the numbers from $a_1 + \ldots + a_n$ to $a_1 + \ldots + a_n + a_{n+1}$ are sums of fome of the numbers a_1, a_2, ..., a_{n+1}. Thus, when n increases by 1, no new gaps occur, and at least one old gap (a_{n+1} itself) disappears, q.e.d.

Therefore at some moment there will be no gaps.

We see that $a_{n+1} = a_1 + \ldots + a_n + 1$ for all large enough n. Then $a_{n+2} = a_1 + \ldots + a_n + a_{n+1} + 1 = (a_1 + \ldots + a_n + 1) + a_{n+1} = 2a_{n+1}$, q.e.d.

5. Let $C(k)$ denotes the sum of all different prime divisors of a positive integer k. For example, $C(1) = 0$, $C(2) = 2$, $C(45) = 8$. Find all positive integers n such that $C(2^n + 1) = C(n)$.

The answer is $n = 3$.

Solution. Let $P(t)$ be the largest prime divisor of a positive integer $t > 1$.

Let m be the largest odd divisor of n: $n = 2^k m$. Then $2^n + 1 = 2^{2^k m} + 1 = a^m + 1$, where $a = 2^{2^k}$. If $k > 0$, that is, n is even, then $C(n) = C(m) + 2$ and $C(2^n + 1) = C(a^m + 1)$.

We need the following two lemmas.

Lemma 1. For every prime $p > 2$ we have $P\left(\frac{a^p+1}{a+1}\right) = p$ or $P\left(\frac{a^p+1}{a+1}\right) \geqslant 2p + 1$.

Proof. Let $P\left(\frac{a^p+1}{a+1}\right) = q$. It follows from Fermat's little theorem that q divides $2^{q-1} - 1$ and therefore $(a^{2p} - 1, a^{q-1} - 1) = a^{(2p, q-1)} - 1$. The greatest common divisor $(2p, q - 1)$ is even and must equal $2p$ or 2. In the first case $2p$ divides $q - 1$, whence $q \geqslant 2p + 1$. In the second case q divides $a^2 - 1$ but not $a - 1$ (because $a^p + 1$ is divisible by q), that is, $a \equiv -1 \pmod q$. Then $\frac{a^p+1}{a+1} = a^{p-1} - \ldots + 1 \equiv p \pmod q$ and $p = q$.

Lemma 2. If p_1 and p_2 are different odd primes then $P\left(\frac{a^{p_1}+1}{a+1}\right) \neq P\left(\frac{a^{p_2}+1}{a+1}\right)$.

Proof. If $P\left(\frac{a^{p_1}+1}{a+1}\right) = P\left(\frac{a^{p_2}+1}{a+1}\right) = q$ then q divides $a^{2p_1} - 1$ and $a^{2p_2} - 1$, therefore $(a^{2p_1} - 1, a^{2p_2} - 1) = a^{(2p_1, 2p_2)} - 1 = a^2 - 1$ and hence $a + 1$, but then $p_1 = q$ and $p_2 = q$, a contradiction.

We are ready now to slve the problem. Let p_1, ..., p_s be all the prime divisors of n. It follows from Lemma 2 that

$$C(2^n + 1) \geqslant P\left(\frac{a^{p_1} + 1}{a + 1}\right) + \ldots + P\left(\frac{a^{p_s} + 1}{a + 1}\right).$$

If $C(2^n + 1) > P\left(\frac{a^{p_1}+1}{a+1}\right) + \ldots + P\left(\frac{a^{p_s}+1}{a+1}\right)$, then $2^n + 1$ has at least one prime divisor not summed in the L.H.S., that is,

$$C(2^n + 1) \geqslant P\left(\frac{a^{p_1} + 1}{a + 1}\right) + \ldots + P\left(\frac{a^{p_s} + 1}{a + 1}\right) + 3 \geqslant p_1 + \ldots + p_s + 3 > C(n).$$

Therefore we can assume the equality:

$$C(2^n + 1) = P\left(\frac{a^{p_1} + 1}{a + 1}\right) + \ldots + P\left(\frac{a^{p_s} + 1}{a + 1}\right).$$

If in this case there is an i such that $P\left(\frac{a^{p_i}+1}{a+1}\right) \neq p_i$, then $C(2^n + 1) \geqslant p_1 + \ldots + p_s + p_i + 1 > C(n)$.

It remains to consider the case when $P\left(\frac{a^{p_i}+1}{a+1}\right) = p_i$ for all i. In this case we have $C(n) = C(2^n + 1) = p_1 + \ldots + p_s$, so n must be odd and $a = 2$. But $2^p \equiv 2 \pmod p$ for all odd prime p, therefore $p > 3$ cannot divide $2^p + 1$. Thus $s = 1$, $p = 3$, $n = 3^r$ with some positive integral r. The number $2^n + 1 = 2^{3^r} + 1$ must be a power of 3. However 19 divides this number for $r = 2$ and consequently for all $r \geqslant 2$. Thus the only remaining case is $n = 3$, which obviously satisfies the condition.

6. A regular tetrahedron $ABCD$ and points M, N are given in space. Prove the inequality

$$MA \cdot NA + MB \cdot NB + MC \cdot NC \geqslant MD \cdot ND.$$

(A tetrahedron is called *regular* if all its six edges are equal.)

Solution. We need the following

Lemma 1. For every different points A, B, C, D the inequality

$$AB \cdot CD + BC \cdot AD \geqslant AC \cdot BD$$

holds.

Proof. Consider the point A_1 on the ray DA such that $DA_1 = \frac{1}{DA}$. In the same way we take the points B_1 and C_1 on the rays DB and DC. Since $\frac{DA_1}{DB} = \frac{DB_1}{DA} = \frac{1}{DA \cdot DB}$, it follows from similarity of the triangles DAB and DB_1A_1 that $A_1B_1 = \frac{AB}{DA \cdot DB}$. Similarly, $B_1C_1 = \frac{BC}{DB \cdot DC}$ and $C_1A_1 = \frac{CA}{DC \cdot DA}$ (1). Substituting these equalities in the triangle inequality $A_1B_1 + B_1C_1 \geqslant A_1C_1$ we obtain $AB \cdot CD + BC \cdot AD \geqslant AC \cdot BD$.

Lemma 2. For every points M, N in the plane of the triangle ABC

$$\frac{AM \cdot AN}{AB \cdot AC} + \frac{BM \cdot BN}{BA \cdot BC} + \frac{CM \cdot CN}{CA \cdot CB} \geqslant 1. \qquad (*)$$

Proof. In the plane ABC we consider the point K such that $\angle ABM = \angle KBC$, $\angle MAB = \angle CKB$. Note that

$$\frac{CK}{BK} = \frac{AM}{AB}, \frac{AK}{BK} = \frac{CM}{BC}, \frac{BC}{BK} = \frac{BM}{AB}. \qquad (2)$$

Applying lemma 1 to the points A, N, C, K we have $AN \cdot CK + CN \cdot AK \geqslant AC \cdot NK$. Triangle inequality $NK \geqslant BK - BN$ gives us $AN \cdot CK + CN \cdot AK \geqslant AC \cdot (BK - BN)$. Hence we obtain

$$\frac{AN \cdot CK}{AC \cdot BK} + \frac{CN \cdot AK}{AC \cdot BK} + \frac{BN}{BK} \geqslant 1. \qquad (3)$$

It follows from (3) and (2) that $\frac{AM \cdot AN}{AB \cdot AC} + \frac{BM \cdot BN}{BA \cdot BC} + \frac{CM \cdot CN}{CA \cdot CB} \geqslant 1$.

Corollary. The inequality (*) remains true when one of the points M, N, or both, lie outside the plane of the triangle ABC.

It follows from lemma 2 when instead of M and N it is applied to their projections onto the plane of the triangle ABC.

We are ready now to solve the problem. On the ray DA we consider the point A_1 such that $DA_1 = \frac{1}{DA}$. In a similar way we take points B_1, C_1, M_1, N_1 on the rays DB, DC, DM, DN.

Applying the corollary of Lemma 2 to the points M_1, N_1 and the triangle $A_1B_1C_1$ we get the inequality $A_1M_1 \cdot A_1N_1 + B_1M_1 \cdot B_1N_1 + C_1M_1 \cdot C_1N_1 \geqslant A_1B_1^2$; using equations similar to (1) we obtain

$$\frac{AM}{DA \cdot DM} \cdot \frac{AN}{DA \cdot DN} + \frac{BM}{DB \cdot DM} \cdot \frac{BN}{DB \cdot DN} + \frac{CM}{DC \cdot DM} \cdot \frac{CN}{DC \cdot DN} \geqslant \left(\frac{AB}{DA \cdot DB}\right)^2,$$

whence

$$AM \cdot AN + BM \cdot BN + CM \cdot CN \geqslant DM \cdot DN,$$

q.e.d.