1. A non-isosceles acute triangle ABC is inscribed in a circle ω. Let H be the orthocentre of this triangle and M the midpoint of AB. Points P and Q on the arc AB of the circle ω that does not contain C satisfy $\angle ACP = \angle BCQ < \angle ACQ$. Let R and S be the feet of the perpendiculars from H onto CQ and CP respectively. Prove that P, Q, R, S lie on the same circle, and M is the centre of this circle.

Solution. Let AA_1, BB_1, CC_1 be the altitudes of the triangle ABC, and L the point where CC_1 meets PQ. Without loss of generality we assume that H lies inside the angle ACQ. Note that the points C, A_1, B_1, R, S, H belong to the circle with diameter CH. Obviously $PQ \parallel AB$. Then $\angle HLQ = 90°$. It follows that H, R, Q, L belong to the circle with diameter HQ. Therefore $\angle CSR = \angle CHR = \angle RQL = \angle RQP$, that is, P, Q, R, S are concyclic.

Since $\angle AA_1B = \angle BB_1A = 90°$, we have $MA_1 = MB_1 = AB/2$. Thus M lies on the perpendicular bisector of A_1B_1.

It follows from the equalities $\angle A_1CR = \angle BCQ = \angle ACP = \angle B_1CS$ that A_1B_1SR is an isosceles trapezoid. Therefore perpendicular bisectors of A_1B_1 and RS coincide. Then M lies on the perpendicular bisector of RS. On the other hand, $APQB$ is an isosceles trapezoid, so M also belongs to the perpendicular bisector of PQ. This means that M is the centre of the circle containing P, Q, R, S.

2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $(x + y^2)f(yf(x)) = xyf(y^2 + f(x))$ for all real x, y.

Answer: $f(x) = x$; $f(x) \equiv 0$; $f(x) = \begin{cases} 0, x \neq -a^2, \\ a, x = -a^2 \end{cases}$ for arbitrary $a \in (-\infty, -1] \cup (0, +\infty)$.

Solution: Setting $y = 0$ in the given equation

$$(x + y^2)f(yf(x)) = xyf(y^2 + f(x)) \tag{1}$$

we have $f(0) = 0$. Now changing y by $-y$ in (1) we obtain $(x+y^2)f(-yf(x)) = -xyf(y^2+f(x)) = -(x + y^2)f(yf(x))$ whence

$$(x + y^2)\left(f(yf(x)) + f(-yf(x))\right) = 0 \tag{2}$$

for all $x, y \in \mathbb{R}$.

Denote $A = \{x|f(x) = 0\}$. Consider the following cases:

(i) $A = \{0\}$. Set $x = -y^2$ in (1), then $-y^3f(y^2 + f(-y^2)) = 0$ whence $y^2 + f(-y^2) = 0$, i.e. $f(t) = t$ for all $t \leqslant 0$. Now for $x = 1$ the equation (2) gives $f(-yf(1)) = -f(yf(1))$, or (because $f(1) \neq 0$) $f(-x) = -f(x)$ for all x. Therefore $f(x) = x$ for all real x. It is easy to see that this function satisfies the given equation.

(ii) $A = \mathbb{R}$, then $f(x) = 0$ for all x is obviously a solution of (1).

(iii) Now let $A \neq \{0\}$, $A \neq \mathbb{R}$. That is, there exist $b \neq 0$, $d \neq 0$ such that $f(b) = 0$ and $f(d) \neq 0$; denote $f(d) = a$. Set $x = b$ in (1) thus obtaining $yf(y^2) = 0$, i.e. $f(t) = 0$ for all $t \geqslant 0$. Now set $x = d$ in (2). Then $f(ay) + f(-ay) = 0$ if $y^2 + d \neq 0$. One of $f(ay)$, $f(-ay)$ is zero since $f(t) = 0$ for $t \geqslant 0$. Hence for $d > 0$ we have $f(ay) = f(-ay) = 0$ for all y, contrary to $f(d) \neq 0$. Thus $d < 0$ and the only possible nonzero value of the function f can be $f(\pm ay)$ for y satisfying $y^2 + d = 0$, i.e. $f(\pm a\sqrt{-d})$. Together with $f(d) \neq 0$ this gives $d = \pm a\sqrt{-d}$, so $d = -a^2$.

It remains to check whether the function given by $f(x) = \begin{cases} 0, x \neq -a^2, \\ a, x = -a^2 \end{cases}$ does satisfy the equation. If $x \neq -a^2$ then (1) becomes $(x+y^2)f(y\cdot 0) = xyf(y^2)$, or $0 = 0$. Now let $x = -a^2$, then we need to check whether the equality $(y^2 - a^2)f(ay) = -a^2yf(y^2 + a)$ holds for all y. Note that left-hand side of the equality is zero anyway ($f(ay) = 0$ for $y \neq -a$ and $y^2 - a^2 = 0$ for $y = -a$). Thus we need $yf(y^2 + a) = 0$ for all y. For this to be valid the equation $y^2 + a = -a^2$ should not have a real solution $y \neq 0$. Thus $-a^2 - a \geqslant 0$, so $a \in (-\infty, -1] \cup (0, +\infty)$ (we remind that $a \neq 0$).

3. A rectangle on a squared paper with 1×1 squares is divided into domino figures (that is, rectangles made of two unit squares sharing a side). Prove that all the vertices of squares inside the rectangle and on its border can be coloured in three colours so that the following condition is

satisfied for each two vertices with distance 1: they have different colours if the segment connecting them lies on the border of a domino figure, and same colour otherwise.

Solution. We prove that there is a desired colouring of a special kind. Let us assign the numbers 1, 2, 3 to the colours and call a square *right* if these colours are found in this order when we move around the square clockwise, and *left* otherwise. Suppose the squares of the rectangle are coloured chequerwise, so that in every domino figure one square is black and another white. Now we claim there is a desired colouring of vertices such that all the black squares are right and all the white squares are left; we call such a colouring *regular*.

Note that the regular colouring of a square in our rectangle is defined by the colour of any one of its vertices. Indeed, if the square is, say, black (so it must be right according to the colouring of its vertices), we can start at the vertex whose colour is known and move clockwise, adding $1 \mod 3$ to the colour number when moving along the domino side and leaving colour unchanged when not.

If two squares share a side, starting this procedure from one of their common vertices gives the same colour for the other one: one of the squares is black, and another is white, and moving from one common vertex to another is clockwise in one square and counterclockwise in the other. Therefore if a square having one still uncoloured vertex shares sides with two squares whose vertices are already coloured, we can colour the remaining vertex so that that the square is coloured regularly. If a square with two still uncoloured vertices shares a side with a square whose vertices are regularly coloured, we can obviously colour its remaining vertices regularly.

Applying this procedure we can colour the vertices of all squares of the rectangle. First we define which squares must be left and which right. Then we colour the vertices of the left lower square, choosing the colour of the left lower vertex arbitrarily. After that we can successively colour the vertices of the squares in the lower row. Then every next row can be coloured starting from the left square: the leftmost square shares a side with only one square whose vertices are already coloured, and every next square with two.