4. The Crocodile thought of four unit squares of a 2018×2018 forming a rectangle with sides 1 and 4. The Bear can choose any square formed by 9 unit squares and ask whether it contains at least one of the four Crocodile's squares. What minimum number of questions should he ask to be sure of at least one affirmative answer?

The answer is $\frac{673^2-1}{2} = 226464$.

Solution. We call *checked* any square chosen by the Bear, and all its unit squares. The position of a unit square in the table can be defined by the numbers of its row and column, that is, the square (x, y) is in the x-th row and y-th column.

First we prove that $\frac{673^2-1}{2}$ questions is enough even on a 2019×2019 table. Let us divide this table into 3×3 squares and apply chess colouring to these large squares so that the corners are white. Thet it is enough to check all the black 3×3 squares: no row or column contains four consecutive white squares.

To prove that we need so many questions, we select all the unit squares with coordinates $(3m+1, 3n+1)$, where $0 \leqslant m, n \leqslant 672$. A 3×3 square obviously can not contain two selected unit squares. On the other hand, if two selected squares lie at distance 3 (i.e., one of them is (x, y), and another is $(x, y+3)$ or $(x+3, y)$), the Bear must check at least one of these two squares (because if neither is checked, then so are the two unit squares between them, and the Crocodile can place his rectangle on the unchecked squares).

Thus it is enough to produce $\frac{673^2-1}{2}$ pairs of selected unit squares at distance 3. One can take pairs $(6k+1, 3n+1)$, $(6k+4, 3n+1)$, $0 \leqslant k \leqslant 335$, $0 \leqslant n \leqslant 672$, and $(2017, 6n+1)$, $(2017, 6n+4)$, $0 \leqslant n \leqslant 335$.

5. Find all real a for which there exists a function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x - f(y)) = f(x) + a[y]$ for every real x и y ($[y]$ denotes the integral part of y).

Answer: $a = -n^2$ for arbitrary integer n.

Solution. First note that $a = 0$ satisfies the problem condition (for example, the equation is satisfied by the function $f(x) \equiv 0$).

Now suppose $a \neq 0$.

Lemma. $f(y) = f(z)$ if and only if $[y] = [z]$.

Suppose $f(y) = f(z)$ for some y, z. Then the given equation implies $f(x)+a[y] = f(x-f(y)) = f(x-f(z)) = f(x)+a[z]$ whence $[y] = [z]$. Conversely, if $[y] = [z]$ then $f(x - f(y)) = f(x) + a[y] = f(x) + a[z = f(x - f(z))]$. It follows from previous observation that $[x - f(y)] = [x - f(z)]$ for all x. Set $x = \frac{f(y)+f(z)}{2}$, then $\left[\frac{f(y)-f(z)}{2}\right] = \left[-\frac{f(y)-f(z)}{2}\right]$, so $f(y) = f(z)$. The lemma is proved.

Now we claim that $f(m) \in \mathbb{Z}$ for any $m \in \mathbb{Z}$. Setting $y = m$ in the given equation we obtain $f(x - f(m)) = f(x) + am$ for any $m \in \mathbb{Z}$, $x \in \mathbb{R}$. Suppose that $f(m) \notin \mathbb{Z}$ for some $m \in \mathbb{Z}$. Choose $t \in (0, 1)$ such that $[f(m)] = [f(m) + t]$. Then for $x = 0$ we have $f(-f(m)) = f(0) + am$ and for $x = -t$ we have $f(-t - f(m)) = f(-t) + am$. Using the lemma we have $f(-f(m)) = f(-t - f(m))$, so $f(0) = f(-t) = f(-1)$, which contradicts the lemma.

From now on we will use in the given equation $f(x - f(y)) = f(x) + ay$ (1) only integer numbers x, y. Setting $y = 1$ in (1) we obtain that $a \in \mathbb{Z}$. Further, for $y = 0$ we have $f(x - f(0)) = f(x)$ and therefore $x - f(0) = x$ (by lemma), whence $f(0) = 0$. Now set $x = f(y)$, then $f(f(y)) = -ay$ (2); replacing y by $f(y)$ in (1) we get $f(x + ay) = f(x) + af(y)$ (3). Denoting $f(1)$ by n and setting $y = 1$ in (3) we obtain $f(x + a) = f(x) + an$ (4). Applying (4) to $x = 0$ we get $f(a) = an$. From (4) we easily conclude that $f(ka) = kan$ for any $k \in \mathbb{Z}$; in particular $f(an) = an^2$. Now setting $y = a$ in (2) gives $-a^2 = f(f(a)) = an^2$ as stated.

It remains to note that if $a = -n^2$ then the function $f(x) = n[x]$ satisfies the given condition: $n[x - n[y]] = n[x] - n^2[y]$, which is obvious.

6. A convex hexagon $ABCDEF$ is inscribed in a circle with radius R. Diagonals AD and BE, BE and CF, AD and CF of the hexagon meet at points M, N, K respectively. Let r_1, r_2, r_3, r_4, r_5, r_6 be the inradii of the triangles ABM, BCN, CDK, DEM, EFN, AFK respectively. Prove that $r_1 + r_2 + r_3 + r_4 + r_5 + r_6 \leqslant R\sqrt{3}$.

Solution.

We start with a lemma.

Lemma. Let R be the circumradius of a quadrilateral $XYZT$, the diagonals of $XYZT$ meet at U, and $\varphi = \frac{1}{2}\angle XUY$. Then the radii r_1 and r_2 of the incentres of XYU and ZTU satisfy

$$\frac{r_1 + r_2}{R} \leqslant 2\tan\varphi(1 - \sin\varphi). \qquad (1)$$

Indeed, let $\angle UXY = 2\psi$, $\angle UYX = 2\vartheta$, then $\angle UTZ = \angle UXY = 2\psi$, $\angle UZT = \angle UYX = 2\vartheta$ (and obviously $\psi + \vartheta + \varphi = \frac{\pi}{2}$). We have $XY + ZT = (r_1 + r_2)(\cot\psi + \cot\vartheta) = 2R\sin\angle XTY + 2R\sin(2\varphi - \angle XTY) = 2R(\sin\angle XTY + \sin(2\varphi - \angle XTY) = 2R \cdot 2\sin\varphi\cos(\varphi - \angle XTY) \leqslant 4R\sin\varphi$. Therefore

$$\frac{r_1 + r_2}{R} \leqslant \frac{4\sin\varphi}{\cot\psi + \cot\vartheta} = \frac{4\sin\varphi\sin\psi\sin\vartheta}{\sin(\psi + \vartheta)} = \frac{4\sin\varphi\sin\psi\sin\vartheta}{\cos\varphi} = 4\tan\varphi\sin\psi\sin\vartheta =$$

$$= 4\tan\varphi \cdot \frac{1}{2}(\cos(\psi - \vartheta) - \cos(\psi + \vartheta)) \leqslant 2\tan\varphi(1 - \sin\varphi),$$

q.e.d.

Returning to the problem, let $\angle AMB = 2\alpha$, $\angle BNC = 2\beta$, $\angle CKD = 2\gamma$, then $\alpha + \beta + \gamma = \frac{\pi}{2}$.

Applying the inequality (1) to the quadrilaterals $ABDE$, $BCEF$ и $CDFA$ we get

$$\frac{r_1 + r_2 + r_3 + r_4 + r_5 + r_6}{R} = \frac{r_1 + r_4}{R} + \frac{r_2 + r_5}{R} + \frac{r_3 + r_6}{R} \leqslant 2\tan\alpha(1 - \sin\alpha) + 2\tan\beta(1 - \sin\beta) + 2\tan\gamma(1 - \sin\gamma).$$

We claim that if $\alpha+\beta+\gamma=\frac{\pi}{2}$ then

$$2\tan\alpha(1-\sin\alpha)+2\tan\beta(1-\sin\beta)+2\tan\gamma(1-\sin\gamma)\leqslant\sqrt{3}. \tag{2}$$

To prove that we consider the function $f(x)=2\tan x(1-\sin x)$ for $x\in(0;\frac{\pi}{2})$.

Since $f''(x)=-2\frac{(1-\sin x)^2+\cos^4 x}{\cos^3 x}<0$ for $x\in(0;\frac{\pi}{2})$, it follows from Jensen's inequality that

$$f(\alpha)+f(\beta)+f(\gamma)\leqslant 3f\left(\frac{\alpha+\beta+\gamma}{3}\right)=3f(\frac{\pi}{6})=\sqrt{3}.$$

Thus (2) is proved, and $r_1+r_2+r_3+r_4+r_5+r_6\leqslant\sqrt{3}R$.