## Solutions for 1-st round IZhO 2018

1. Let $\alpha$, $\beta$, $\gamma$ be the angles of a triangle opposite to the sides $a$, $b$, $c$ respectively. Prove the inequality

$$2\left(\cos^2\alpha+\cos^2\beta+\cos^2\gamma\right)\geq \frac{a^2}{b^2+c^2}+\frac{b^2}{a^2+c^2}+\frac{c^2}{a^2+b^2}.$$

**Solution.** By the Law of Sines, RHS equals $\frac{\sin^2\alpha}{\sin^2\beta+\sin^2\gamma}+\frac{\sin^2\beta}{\sin^2\alpha+\sin^2\gamma}+\frac{\sin^2\gamma}{\sin^2\alpha+\sin^2\beta}$.
Applying Cauchy-Bunyakowski inequality we have

$$\sin^2\alpha=\sin^2(\beta+\gamma)=(\sin\beta\cos\gamma+\sin\gamma\cos\beta)^2\leq(\sin^2\beta+\sin^2\gamma)(\cos^2\gamma+\cos^2\beta),$$

therefore $\cos^2\beta+\cos^2\gamma\geq\frac{\sin^2\alpha}{\sin^2\beta+\sin^2\gamma}$.
Adding similar inequalities for $\cos^2\gamma+\cos^2\alpha$ and $\cos^2\alpha+\cos^2\beta$ we get the desired result.

2. Points $N$, $K$, $L$ lie on the sides $AB$, $BC$, $CA$ of a triangle $ABC$ respectively so that $AL=BK$ and $CN$ is the bisector of the angle $C$. The segments $AK$ and $BL$ meet at the point $P$. Let $I$ and $J$ be the incentres of the triangles $APL$ and $BPK$ respectively. The lines $CN$ and $IJ$ meet at point $Q$. Prove that $IP=JQ$.

**Solution.** The case $CA=CB$ is trivial. If $CA\neq CB$, we may suppose, without loss of generality, that $CN$ meets the segment $PK$.

Let the circumcircles $\omega_1$ and $\omega_2$ of the triangles $APL$ and $BPK$ respectively meet again at point $T$. Then

$$\angle LAT=\angle TPB=\angle TKB. \qquad (1)$$

and $\angle ALT=\angle APT=\angle TBK$, that is, $\triangle ALT=\triangle KBT$, hence

$$AT=TK. \qquad (2)$$

It follows from (1) that the quadrilateral $ACKT$ is cyclic; together with (2) this means that $\angle ACT=\angle TCK$, i.e. $T$ lies on the bisector of $CN$.

Let $IJ$ meet $\omega_1$ and $\omega_2$ at $I_1$ and $J_1$ respectively. Since $\omega_1$ and $\omega_2$ have equal radii and $AL=BK$, the triangles $ALI_1$ and $BKJ_1$ are equal. We use Mansion's lemma: the midpoint of arc $XY$ of the circumcircle of $XYZ$ lies at equal distances from the ends of this arc and the incentre. It follows from this lemma that $I_1I=I_1L=J_1K=J_1J$. Moreover, $\angle PI_1T=\angle PAT=\angle PKT=\angle PJ_1T$, therefore, $I_1T=J_1T$. Thus $T$ lies on the median bisector of $I_1J_1$ and on the median bisector of $IJ$.

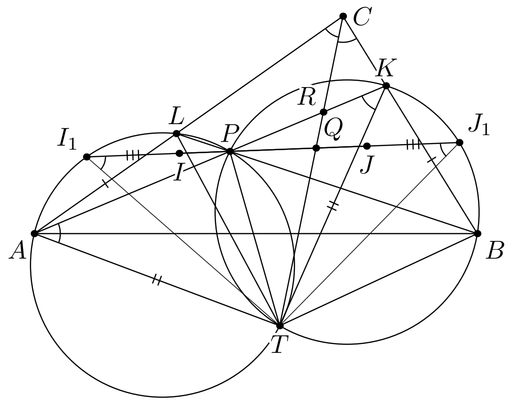


Figure 1: image

It remains to prove that $T$ lies on the median bisector of $PQ$. Let $R=AK\cap CT$. Then $\angle ART=\angle RAC+\angle ACR=\angle RAC+\angle AKT=\angle RAC+\angle KAT=\angle LAT=\angle BPT$. Since $PQ$ bisects the angle $RPB$, $\angle PQT=\angle PRT+\angle RPQ=\angle PBT+\angle BPJ=\angle TPQ$, therefore $T$ belongs to the median bisector of $PQ$ and $IP=JQ$.

3. Prove that there exist infinitely many pairs $(m,n)$ of positive integers such that $m+n$ divides $(m!)^n+(n!)^m+1$.

**Solution.** We shall find a pair such that $m+n=p$ is prime and $n$ is even. Applying Wilson's theorem we have

$$m!=(p-n)!=\frac{(p-1)!}{(p-n+1)\dots(p-2)(p-1)}\equiv\frac{-1}{-(n-1)\dots(-2)(-1)}\equiv\frac{1}{(n-1)!}\equiv\frac{n}{n!}\pmod p.$$

It follows from Fermat's Little Theorem that $(n!)^p\equiv n! \pmod p$, therefore

$$(m!)^n+(n!)^m+1\equiv\left(\frac{n}{n!}\right)^n+(n!)^{p-n}+1\equiv\frac{n^n+n!+(n!)^n}{(n!)^n}\pmod p;$$

thus it suffices to prove that the number $n^n+n!+(n!)^n$ has a prime divisor $p>n$ for infinitely many even $n$.

We prove that this condition is satisfied, for instance, by all the numbers of the form $n=2q$, where $q>2$ is prime. Let $A=(2q)^{2q}+(2q)!+((2q)!)^{2q}$. For a prime $p$ and integer $k$ we denote by $v_p(k)$ the largest integer $\ell$ such that $p^\ell$ divides $k$.

If $r<2q$ is prime and $r\notin\{2,q\}$ then $A\equiv(2q)^{2q}\not\equiv 0 \pmod r$. The largest degree of $q$ dividing $(2q)!$ is $q^2$, while for $(2q)^{2q}$ and $((2q)!)^{2q}$ it is $2q$ and $4q$ respectively, therefore $v_q(A)=2$.

Finally, $v_2((2q)!)=\left[\frac{2q}{2}\right]+\left[\frac{2q}{4}\right]+\left[\frac{2q}{8}\right]+\cdots<\frac{2q}{2}+\frac{2q}{4}+\frac{2q}{8}+\cdots=2q$, so $v_2((2q)!)<v_2((2q)^{2q})$ and obviously $v_2((2q)!)<v_2((2q)!^{2q})$, thus $v_2(A)\leq 2q-1$. On the other hand, $A>(2q)^{2q}>2^{2q-1}q^2$, therefore $A$ has a prime divisor $p>2q$, q.e.d.