1. Each integral point of the plane is coloured white or blue. Prove that one can choose a colour so that for every positive integer n there is a triangle of area n with three vertices of the chosen colour.

Solution. If every two neighbouring points (that is, points at distance 1) have different colours then, in fact, we have a monochromatic lattice of $\sqrt{2} \times \sqrt{2}$ squares, where triangle with any integral area is easily found.

When this is not the case, we consider neighbouring points A and B ($AB = 1$) of the same colour (say white). To find a triangle of area n, we need a white point on the line parallel to AB at the distance $2n$. If there is such point for each n, we are done. Otherwise, we have a line ℓ with blue points only.

Consider a line ℓ_1 "next to ℓ", that is, the line parallel to ℓ at distance 1 from it. If it contains a blue point, we have a triangle with blue vertices of area $\frac{n}{2}$ for each positive integer n.

The only remaining case is that of line ℓ_1 containing only white points. Then we consider the line $\ell_2 \neq \ell$ at distance 1 from ℓ_1, and again, if there is a white point on ℓ_2, we are done. Now, if all points of ℓ_2 are blue, then for each n we have a triangle of area n with three blue vertices.

2. Inside the triangle ABC a point M is given. The line BM meets the side AC at N. The point K is symmetrical to M with respect to AC. The line BK meets AC at P. If $\angle AMP = \angle CMN$, prove that $\angle ABP = \angle CBN$.

Solution. Let D, E, F be the feet of perpendiculars to BP, MP, BM respectively drawn through A, and G, Q, H be the feet of perpendiculars to BP, MP, BM respectively drawn through C.

Note that $\triangle AFM \sim \triangle CQM$ and $\triangle AME \sim \triangle CMH$, therefore $\frac{AF}{CQ} = \frac{AM}{CM} = \frac{AE}{CH}$. By symmetry we have also $CQ = CG$, $AE = AD$ and $\angle FAD = \angle FBD = \angle GCH$, therefore $\frac{AF}{CG} = \frac{AD}{CH}$. It follows that $\triangle FAD \sim \triangle GCH$, thus $\angle AFD = \angle CGH$.

Now the points A, B, F, D are concyclic, therefore $\angle ABP = \angle AFD$, and similarly $\angle CBN = \angle CGH$. Combining that with the above, we have $\angle ABP = \angle CBN$.

Problem 3

Setting $x = 1$, $y = 0$ in the initial equation

$$f(x^3 + y^3 + xy) = x^2 f(x) + y^2 f(y) + f(xy) \qquad (1)$$

gives $f(0) = 0$.

Taking $y = 0$ in (1) we obtain

$$f(x^3) = x^2 f(x). \qquad (2)$$

Substituting $y = -x$ into (1) leads to

$$f(-x^2) = x^2 f(x) + x^2 f(-x) + f(-x^2) \Rightarrow f(-x) = -f(x). \qquad (3)$$

From (1) and (3) it follows that

$$\begin{aligned} &f(x^3 + y^3 + xy) + f(x^3 - y^3 - xy) \\ &\quad = x^2 f(x) + y^2 f(y) + f(xy) + x^2 f(x) - y^2 f(y) - f(xy) = 2x^2 f(x) = 2f(x^3). \quad (4) \end{aligned}$$

Note that for any a, $b \in \mathbb{R}$ there exist x, $y \in \mathbb{R}$ such that

$$a = x^3 + y^3 + xy, \quad b = x^3 - y^3 - xy.$$

To this end, we take x, y that satisfy the equations

$$x^3 = \frac{a+b}{2}, \quad y^3 + xy = \frac{a-b}{2}$$

(we see that functions in left hand sides of the equations have the ranges $\mathbb{R}$). Therefore, we can rewrite (4) in the form

$$f(a) + f(b) = 2f\Big(\frac{a+b}{2}\Big), \quad a,\ b \in \mathbb{R}.$$

Thus, we have

$$f(0) + f(a+b) = 2f\Big(\frac{a+b}{2}\Big) \Rightarrow f(a+b) = f(a) + f(b), \quad a,\ b \in \mathbb{R}.$$

Further, we change $x \to x+1$ in (2), denote $c = f(1)$, and from additivity of f obtain

$$\begin{aligned} f((x+1)^3) &= (x+1)^2 f(x+1) \\ \Leftrightarrow f(x^3) + 3f(x^2) + 3f(x) + c &= (x^2 + 2x + 1)(f(x) + c) \\ \Leftrightarrow 3f(x^2) &= (2x - 2)f(x) + (x^2 + 2x)c \qquad (5) \end{aligned}$$

Substituting $x \to -x$ in (5), we get

$$3f(x^2) = (-2x - 2)f(-x) + (x^2 - 2x)c = (2x + 2)f(x) + (x^2 - 2x)c \qquad (6)$$

From the equality of right hand sides of (5) and (6) we obtain

$$f(x) = cx.$$

It is easy to verify that this function satisfies the given equation for all $c \in \mathbb{R}$.

Answer: $f(x) = cx$, $c \in \mathbb{R}$.