4. Determine the maximum integer n such that for each $k \leqslant \frac{n}{2}$ there are two positive divisors of n with difference k.

Solution. The answer is 24. This number obviously satisfies the condition: $1 = 2 - 1$, $2 = 4 - 2$, $3 = 6 - 3$, $4 = 8 - 4$, $5 = 8 - 3$, $6 = 8 - 2$, $7 = 8 - 1$, $8 = 12 - 4$, $9 = 12 - 3$, $10 = 12 - 2$, $11 - 12 - 1$, $12 = 24 - 12$.

Suppose $n > 24$ satisfies the condition. If n is odd, it has no divisors between n and $\frac{n}{3}$, therefore $\frac{n-1}{2}$ must have the form $n - d$, where d divides n. But then $d = \frac{n+1}{2}$ clearly does not divide n. Thus n is even.

If $\frac{n}{3} \leqslant k < \frac{n}{2}$ and $k = d_1 - d_2$, where d_1 and d_2 divide n, then $d_1 = \frac{n}{2}$ (since obviously $d_1 > \frac{n}{3}$, and for $d_1 = n$ the number d_2 must be greater than $\frac{n}{2}$). Therefore, for every such k the number $\frac{n}{2} - k$ divides n. This means that n is divisible by all positive integers not exceeding $\frac{n}{6}$. Since $n > 24$, it is divisible by 3 and 4 and therefore by 12.

The numbers $\frac{n}{6}$ and $\frac{n}{6} - 1$ are coprime and divide n. Therefore their product also divides n, and $n \geqslant \frac{n}{6}\left(\frac{n}{6} - 1\right)$, that is, $n \leqslant 42$. Since $12|n$, it remains to check the number 36, which is not divisible by $5 < \frac{36}{6}$ and therefore does not satisfy the condition.

5. Let A_n be the set of partitions of the sequence 1, 2, ..., n into several subsequences such that every two neighbouring terms of each subsequence have different parity, and B_n the set of partitions of the sequence 1, 2, ..., n into several subsequences such that all the terms of each subsequence have the same parity (for example, the partition $\{(1,4,5,8),(2,3),(6,9),(7)\}$ is an element of A_9, and the partition $\{(1,3,5),(2,4),(6)\}$ is an element of B_6).

Prove that for every positive integer n the sets A_n and B_{n+1} contain the same number of elements.

Solution. To prove that $|A_n| = |B_{n+1}|$ we construct a bijection between the two types of partitions.

Let A be a partition of the first type, that is, the elements of each subsequence in A have alternating parities. We map this partition to the partition B defined by the following rule:

Two numbers $x < y$ are adjacent in some subsequence in A if and only if x and $y+1$ are adjacent in some subsequence in B.

For example, the partition $\{(1,4,7,8),(2,5,10),(3,6),(9)\} \in A_{10}$ is mapped to the partition $\{(1,5,11),(2,6),(4,8),(3,7,9),(10)\} \in B_{11}$.

It follows immediately that all the terms of each subsequence in B have the same parity, that is, $B \in B_{n+1}$.

Transforming each pair (x,z) of consecutive terms in any partition $B \in B_{n+1}$ into pair $(x, z-1)$ (where obviously $x < z-1$ and the numbers x and $z-1$ have different parity) we construct the unique $A \in A_n$ which maps to B. Thus our mapping is a bijection.

6. The area of a convex pentagon $ABCDE$ is S, and the circumradii of the triangles ABC, BCD, CDE, DEA, EAB are R_1, R_2, R_3, R_4, R_5. Prove the inequality

$$R_1^4 + R_2^4 + R_3^4 + R_4^4 + R_5^4 \geqslant \frac{4}{5\sin^2 108^\circ} S^2.$$

Solution. First we prove the following

Lemma 1. In a convex n-gon $A_1A_2\ldots A_n$ with area S we have

$$4S \leqslant A_nA_2 \cdot R_1 + A_1A_3 \cdot R_2 + \ldots + A_{n-1}A_1 \cdot R_n,$$

where R_i is the circumradius of the triangle $A_{i-1}A_iA_{i+1}$, $A_0 = A_n$, $A_{n+1} = A_n$.

Let M_i be the midpoint of A_iA_{i+1} for $i = 1, \ldots, n$. For each i we consider the quadrilateral formed by the segments A_iM_i and A_iM_{i-1} and the perpendiculars to this segments drawn through M_i and M_{i-1}, respectively. We claim that these n quadrilateral cover the n-gon. Indeed, let P be a point inside the n-gon. Let PA_k be the minimum among the distances PA_1, PA_2, $\ldots$, PA_n. We have $PA_k \leqslant PA_{k+1}$ and $PA_k \leqslant PA_{k-11}$, therefore P belongs to the n-gon and to each of the two half-planes containing A_k and bounded by the perpendicular bisectors to A_kA_{k+1} and A_kA_{k-11}, that is, to the k-th quadrilateral. To complete the proof it remains to note that the area of the i-th quadrilateral does nor exceed $\frac{1}{2} \cdot \frac{A_{i-1}A_{i+1}}{2} \cdot R_i$.

For our problem it follows that $4S \leqslant 2R_1^2 \sin\angle A_1 + 2R_2^2 \sin\angle A_2 + \ldots + 2R_5^2 \sin\angle A_5$. Applying Cauchy-Buniakowsky inequality, we obtain

$$2S \leqslant R_1^2 \sin\angle A_1 + R_2^2 \sin\angle A_2 + \ldots + R_5^2 \sin\angle A_5 \leqslant \sqrt{(R_1^4 + \ldots + R_5^4)(\sin^2\angle A_1 + \ldots + \sin^2\angle A_5)} \leqslant$$

$$\leqslant \sqrt{5(R_1^4 + \ldots + R_5^4)\sin^2 108^\circ},$$

thus

$$\frac{4S^2}{5\sin^2 108^\circ} \leqslant R_1^4 + R_2^4 + \ldots + R_5^4.$$

In the above inequality we made use of the following

Lemma 2. If α_1, α_2, $\ldots$, α_5 are angles of a convex pentagon, then $\sin^2\alpha_1 + \ldots + \sin^2\alpha_5 \leqslant 5\sin^2 108^\circ$.

The sum in question does not depend on the order of the angles, therefore we may assume $\alpha_1 \leqslant \alpha_2 \leqslant \ldots \leqslant \alpha_5$.

If $\alpha_1 = 108^\circ$, then $\alpha_2 = \ldots = \alpha_5 = 108^\circ$, and the inequality turns to equality.

If $\alpha_1 < 108^\circ$, then $\alpha_5 > 108^\circ$. Note that $\alpha_1 + \alpha_5 < 270^\circ$ (if $\alpha_1 + \alpha_5 \geqslant 270^\circ$, then $\alpha_2 + \alpha_3 + \alpha_4 \leqslant 270^\circ$, therefore $\alpha_2 \leqslant 90^\circ$, a fortiori $\alpha_1 \leqslant 90^\circ$ and thus $\alpha_5 \geqslant 180^\circ$, a contradiction). Then we have

$$\sin^2 108^\circ + \sin^2(\alpha_1 + \alpha_5 - 108^\circ) - \sin^2\alpha_1 - \sin^2\alpha_5 = 2\cos(\alpha_1 + \alpha_5)\sin(\alpha_1 - 108^\circ)\sin(\alpha_5 - 108^\circ) > 0.$$

It means that changing the angles α_1 by 108° and α_5 by $\alpha_1 + \alpha_5 - 108^\circ$ increases the sum of squares of the sines. Iterating this operation, we shall make all the angles equal to 108°, thus proving the inequality.